APPLICATION OF ASSOCIATED POLYNOMIALS TO CIRCUITS AND SYSTEMS

by

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UNITED STATES
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THESIS

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ABSTRACT

In this thesis, the theory of associated polynomials is applied to linear circuits and systems. Associated polynomials are defined and used in the presentation of a computer algorithm suitable for accomplishing a partial fraction expansion including repeated roots. Variations of the coefficients of a polynomial are related to the variation of its roots (and vice-versa) by associated polynomials. These results are used in a root solving process and to express sensitivity coefficients in a new analytical form. Lastly, a theory of compensating parameter adjustments is developed with applications to self-adaptive systems. It is shown that by compensating adjustments of any two parameters in the system which are linearly related to the coefficients of the characteristic equation, adaptive compensation is possible for a real root of the closed loop transfer function only and that for complex roots, the real or imaginary parts may be kept invarient but not both.

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CHAPTER I

Associated Polynomial Theory

1.1: Introduction

This thesis is concerned with the investigation and application of associated polynomials to the analysis and synthesis of linear systems. As is shown, every polynomial of degree n can be used to generate a sequence of n+1 polynomials of decreasing degree called the associated polynomials of the defining polynomial. The associated polynomials depend in a particularly simple way upon the coefficients of the original polynomial. Due to the relationship between the coefficients of the original polynomial, the associated polynomials, and the properties of the original polynomial which can be expressed simply in terms of the associated polynomials, several useful applications and results are possible in network analysis and control theory.

The concept and use of associated polynomials is not new in that these polynomials have appeared in many guises in the literature[1], usually related to some particular application. Associated polynomials first appeared in the literature in the late eighteenth century when Legrange used them to obtain a solution of the homogeneous linear difference equation[2]. Associated polynomials have been called Horner polynomials[1,3] as they were used directly in Horner's method or rule[4]. But the purely mathematical aspects of associated polynomials are not of direct significance to this thesis. The use of associated polynomials in Horner's method is presented separately

in appendix A. What often occurs in engineering applications is not an associated polynomial itself, but rather a number corresponding to an associated polynomial evaluated at a particular root of the original polynomial.

Recently[1], the properties of associated polynomials have been investigated in themselves and new concepts and results introduced. Reference 1 is written in a pure mathematical language with only mathematical formulas as results. The object of this thesis is to demonstrate that associated polynomials have very useful properties and applications in the fields of circuit and systems theory.

This thesis, divided into five chapters, presents the basic associated polynomial theory in chapter one. In chapter two, this basic theory is used to present an original algorithm suitable for computer programming for accomplishing a partial fraction expansion including repeated roots of high order of multiplicity. Two theorems are developed in chapter three that show that the variation of the coefficients of a polynomial with the variation of its roots or vice-versa is determined by associated polynomials. The theorems are made possible by deriving the orthnormality relationship between the set of associated polynomials and the roots of the defining polynomial. The above theorems have important engineering applications in that they are used in a root solving process in chapter three, and in chapter four to express sensitivity coefficients in a new analytical form. Chapter five shows how associated polynomial theory may be used in the development of a theory of compensating parameter adjustments with

application to self-adaptive systems in which a change in a variable parameter in a plant may be compensated for by a change in another variable parameter such that the dominant root performance of the overall system is controlled. An important theorem of this chapter is the proof that if a feedback control system has a real root then the location of this root may be kept constant by compensating adjustments of any two parameters in the system which are linearly related to the coefficients of the characteristic equation. It is shown that when the compensating parameters are associated with simple zeros and poles of the open loop transfer function, such adaptive compensation is possible for a real root of the closed loop transfer function only and that for complex conjugate roots the real or imaginary parts may be kept invariant but not both. This theorem is of fundamental importance in adaptive control theory.

1.2: Definition of Associated Polynomials

Let a monic polynomial, $F(s)$, of degree n be expressed as:

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0 = \sum_{j=0}^{n} a_j s^j \; ; \; a_n = 1 \qquad (1\text{-}1)$$

The condition that $F(s)$ be monic, i.e., $a_n = 1$, is convenient rather than essential. Let the distinct roots of $F(s)$ be labeled, p_j, $j = 1,2,\ldots,n$.

If u is any variable different from s, then the polynomials $A_j(u)$; $j = 0,1,\ldots,n$ are defined by the operation:

$$\frac{sF(s)-uF(u)}{s-u} = A_0(u) + A_1(u)s + \ldots + A_n(u)s^n = \sum_{j=0}^{n} A_j(u)s^j \qquad (1\text{-}2)$$

The set of polynomials, $A_j(u)$; $j = 0,1,\ldots,n$ are called the associated polynomials of F(s).

<u>Example 1-1</u>

Given the polynomial: $F(s) = s^2 + 3s + 2$

The associated polynomials of F(s) are found by the following operations:

$$F(u) = u^2 + 3u + 2$$

Performing the multiplication and division of equation 1-2,

$$\frac{sF(s)-uF(u)}{s-u} = s^2 + (3+u)s + (u^2+3u+2)$$

The associated polynomials of F(s) are:

$$A_0(u) = u^2 + 3u + 2$$

$$A_1(u) = 3 + u$$

$$A_2(u) = 1$$

The basic definition of associated polynomials given by equation 1-2 may be simplified by noting the relationship of associated polynomials and the coefficients of the basic polynomial. This relationship is given by Theorem 1-1.

<u>THEOREM 1-1</u>

The associated polynomials of an n^{-th} order polynomial, F(s), may be found from the coefficients of F(s) by the recursion formula:

$$A_j(u) = \sum_{i=j}^{n} a_i u^{i-j} \; : \; j = 0,1,\ldots,n$$

Proof:

Multiplying equation 1-2 by (s-u);

$$sF(s)-uF(u) = [A_0(u)+A_1(u)s+...+A_n(u)s^n](s-u) \qquad (1\text{-}3)$$

Substituting equation 1-1 for F(s) and multiplying through by (s-u), equation 1-3 becomes:

$$a_n s^{n+1} + a_{n-1}s^n + ... + a_0 s - uF(u) = A_n(u)s^{n+1} +$$
$$[A_{n-1}(u)-uA_n(u)]s^n + ... + [A_0(u)-uA_1(u)]s - uA_0(u)$$
$$(1\text{-}4)$$

Comparing the coefficients of the equal powers of s,

$$a_n = A_n(u) \qquad (1\text{-}5a)$$

$$a_{n-1} = A_{n-1}(u) - uA_n(u) \qquad (1\text{-}5b)$$

$$a_{n-2} = A_{n-2}(u) - uA_{n-1}(u) \qquad (1\text{-}5c)$$

$$\cdots$$

$$a_0 = A_0(u) - uA_1(u) \qquad (1\text{-}5d)$$

Since $a_n = 1$, equation 1-5 may be explicitly expressed as:

$$A_j(u) = uA_{j+1}(u) + a_j \;:\; j = 0,1,...,n-1 \qquad (1\text{-}6)$$

By definition of the set of associated polynomials, $A_j(u)$, j ranges from 0 to n. Hence let:

$$A_{n+1}(u) = 0 \qquad (1\text{-}7)$$

Using equation 1-7, equation 1-6 becomes:

$$A_j(u) = uA_{j+1}(u) + a_j \;:\; j = 0,1,...,n \qquad (1\text{-}8)$$

In order to form an explicit expression for the associated polynomials, note:

$$A_n(u) = a_n \qquad (1\text{-}9a)$$

$$A_{n-1}(u) = uA_n(u) + a_{n-1} = a_{n-1} + a_n u \qquad (1\text{-}9b)$$

$$A_{n-2}(u) = uA_{n-1}(u) + a_{n-2} = a_{n-2} + a_{n-1}u + a_n u^2 \qquad (1\text{-}9c)$$

. . .

$$A_0(u) = uA_1(u) + a_0 = a_0 + a_1 u + \ldots + a_n u^n \qquad (1\text{-}9n)$$

for j = 0:

$$A_o(u) = \sum_{i=0}^{n} a_i u^i \qquad (1\text{-}10a)$$

for j = 1:

$$A_1(u) = \sum_{i=1}^{n} a_i u^{i-1} \qquad (1\text{-}10b)$$

hence:

$$A_j(u) = \sum_{i=j}^{n} a_i u^{i-j} \qquad j = 0,1,\ldots,n \qquad (1\text{-}11)$$

Corollary 1-1

The associated polynomial, $A_j(u)$, is a polynomial of degree $n - j$.

Corollary 1-2

The particular associated polynomial, $A_0(u)$, is always:

$$A_0(u) = F(u) \qquad (1\text{-}12)$$

Example 1-2

Using the results of Theorem 1-1, the associated polynomials of the polynomial, F(s), of example 1-1 may be determined as follows:

$$F(s) = s^2 + 3s + 2 = a_2 s^2 + a_1 s + a_0$$

$$A_j(u) = \sum_{i=j}^{n} a_i u^{i-j} \quad : \; j = 0,1,2$$

The associated polynomials of F(s) are:

$$A_2(u) = a_2 = 1$$

$$A_1(u) = a_1 + a_2 u = 3 + u$$

$$A_0(u) = a_0 + a_1 u + a_2 u^2 = 2 + 3u + u^2$$

which agrees with example 1-1.

1.3: Characteristics of Associated Polynomials

The above section has shown the elementary relationship of associated polynomials to the coefficients of the original polynomial. There are other interesting properties of associated polynomials that can now be considered. These properties include the relationship between associated polynomials and the original polynomial which leads to the concept of synthetic division and the fact that associated polynomials are elementary symmetric functions. These properties are stated in the following theorems.

Theorem 1-2

The relationship between a polynomial, F(s), and its associated polynomials, given any particular root, p_i, of F(s) is:

$$\frac{F(s)}{s-p_i} = \sum_{j=1}^{n} A_j(p_i) s^{j-1}$$

Proof:

For convenience, equation 1-2 is repeated as:

$$(sF(s))[s,u] = A_0(u) + A_1(u)s + \ldots + A_n(u)s^n \quad (1\text{-}2)$$

Equation 1-2 may be rewritten as:

$$(sF(s))[s,u] = A_0(u) + s[A_1(u) + A_2(u)s + \ldots + A_n(u)s^{n-1}] \quad (1\text{-}13)$$

However the expression in the brackets of equation 1-13 may be equated to:

$$\frac{F(s)-F(u)}{s-u} = A_1(u) + A_2(u)s + \ldots + A_n(u)s^{n-1} = \sum_{j=1}^{n} A_j(u)s^{j-1} \quad (1\text{-}14)$$

Since $F(u) = A_0(u)$, equation 1-14 may be rewritten as:

$$\frac{F(s)}{s-u} = \frac{A_0(u)}{s-u} + \sum_{j=1}^{n} A_j(u)s^{j-1} \quad (1\text{-}15)$$

Note that equation 1-15 states that the quotient and remainder after division by a monic linear polynomial may be calculated recursively. As shown in appendix A, the nature of this recursion leads to synthetic division.

Let $u = p_i$, be a root of the polynomial, $F(s)$, then, since $F(p_i) = 0$:

$$\left.\frac{A_0(u)}{s-u}\right|_{u=p_i} = \left.\frac{F(u)}{s-u}\right|_{u=p_i} = \frac{F(p_i)}{s-p_i} = 0 \quad (1\text{-}16)$$

and equation 1-15 may be rewritten as:

$$\frac{F(s)}{s-p_i} = \sum_{j=1}^{n} A_j(p_i)s^{j-1} \quad (1\text{-}17)$$

Theorem 1-2 may be numerically verified by the following example.

Example 1-3

Given:

$$F(s) = s^4 + 10s^3 + 35s^2 + 50s + 24 = (s+1)(s+2)(s+3)(s+4)$$

Let:

$$p_1 = -1;\ p_2 = -2;\ p_3 = -3;\ p_4 = -4.$$

The coefficients of F(s) are:

$$a_0 = 24;\ a_1 = 50;\ a_2 = 35;\ a_3 = 10;\ a_4 = 1.$$

The associated polynomials of F(s) are:

$$A_4(u) = a_n = 1$$

$$A_3(u) = a_3 + a_4u = 10 + u$$

$$A_2(u) = a_2 + a_3u + a_4u^2 = 35 + 10u + u^2$$

$$A_1(u) = a_1 + a_2u + a_3u^2 + a_4u^3 = 50 + 35u + 10u^2 + u^3$$

Note: The particular trivial associated polynomial $A_0(u)$ is no longer required and is not calculated in future examples.

Consider the root, p_2, of F(s), then:

$$A_1(p_2) = 12;\ A_2(p_2) = 19;\ A_3(p_2) = 8;\ A_4(p_2) = 1.$$

Expanding equation 1-17 and evaluating:

$$\frac{F(s)}{s+2} = A_1(p_2) + A_2(p_2)s + A_3(p_2)s^2 + A_4(p_2)s^3$$

$$\frac{F(s)}{s+2} = 12 + 19s + 8s^2 + s^3$$

As a check:

$$(s^3+8s^2+19s+12)(s+2) = s^4 + 10s^3 + 35s^2 + 50s + 24 \quad \text{as given.}$$

Corollary 1-3

Theorem 1-2 developed a relationship between a polynomial, F(s), and its associated polynomials given a particular root of F(s). This relationship may be extended to polynomials with multiple roots so that the following formulas are available.

CASE (1): TWO REPEATED ROOTS

$$\frac{F(s)}{(s-p_i)^2} = A_n(p_i)s^{n-2} + [A_{n-1}(p_i)+p_iA_n(p_i)]s^{n-3} + [A_{n-2}(p_i)+p_iA_{n-1}(p_i)+p_i^2A_n(p_i)]s^{n-4} + \ldots + [A_2(p_i)+p_iA_3(p_i)+\ldots+p_i^{n-2}A_n(p_i)] \tag{1-18}$$

Expressed as an explicit relationship:

$$\frac{F(s)}{(s-p_i)^2} = \sum_{h=0}^{n-2} s^h \sum_{j=h+2}^{n} p_i^{j-(h+2)} A_j(p_i) \tag{1-19}$$

CASE (2): THREE REPEATED ROOTS

$$\frac{F(s)}{(s-p_i)^3} = A_n(p_i)s^{n-3} + [A_{n-1}(p_i)+2p_iA_n(p_i)]s^{n-4} + [A_{n-2}(p_i)+2p_iA_{n-1}(p_i)+3(p_i)^2A_n(p_i)]s^{n-5} + \ldots + [A_3(p_i)+2p_iA_4(p_i)+\ldots+(n-2)(p_i)^{n-3}A_n(p_i)] \tag{1-20}$$

Expressed as an explicit relationship

$$\frac{F(s)}{(s-p_i)^3} = \sum_{h=1}^{n-2} s^{h-1} \sum_{j=h+2}^{n} [j-(h+1)]p_i^{j-(h+2)} A_j(p_i) \tag{1-21}$$

CASE (3): FOUR REPEATED ROOTS

$$\frac{F(s)}{(s-p_i)^4} = A_n(p_i)s^{n-4} + [A_{n-1}(p_i)+3p_iA_n(p_i)]s^{n-5}$$

$$+ [A_{n-2}(p_i)+3p_iA_{n-1}(p_i)+6(p_i)^2A_n(p_i)]s^{n-6}$$

$$+[A_{n-3}(p_i)+3p_iA_{n-2}(p_i)+6(p_i)^2A_{n-1}(p_i)+10(p_i)^3A_n(p_i)]s^{n-7}$$

$$+ \dots + [A_4(p_i)+3p_iA_5(p_i)+\dots+[1+\dots+(n-3)]p_i^{n-4}A_n(p_i)]$$

(1-22)

Expressed as an explicit relationship,

$$\frac{F(s)}{(s-p_i)^4} = \sum_{h=1}^{n-3} s^{h-1} \sum_{j=h+3}^{n} \{1+\dots+[j-(h+2)]\}p_i^{j-(h+3)}A_j(p_i)$$

(1-23)

<u>Example 1-4</u>

Given:

$$F(s) = s^5 + 13s^4 + 67s^3 + 171s^2 + 216s + 108 = (s+2)^2(s+3)^3$$

Let: $p_1 = -2$; $p_2 = -3$.

The associated polynomials of F(s) are:

$$A_5(u) = 1$$

$$A_4(u) = 13 + u$$

$$A_3(u) = 67 + 13u + u^2$$

$$A_2(u) = 171 + 67u + 13u^2 + u^3$$

$$A_1(u) = 216 + 171u + 67u^2 + 13u^3 + u^4$$

Evaluating the associated polynomials at p_1:

$A_5(p_1) = 1$; $A_4(p_1) = 11$; $A_3(p_1) = 45$; $A_2(p_1) = 81$.

Evaluating the associated polynomials at p_2:

$$A_5(p_2) = 1;\ A_4(p_2) = 10;\ A_3(p_2) = 37.$$

Applying equation 1-18:

$$\frac{F(s)}{(s-p_1)^2} = A_5(p_1)s^3 + [A_4(p_1)+p_1A_5(p_1)]s^2 + [A_3(p_1)+p_1A_4(p_1)+$$

$$p_1{}^2A_5(p_1)]s + [A_2(p_1)+p_1A_3(p_1)+p_1{}^2A_4(p_1)+p_1{}^3A_5(p_1)]$$

Evaluating:

$$\frac{F(s)}{(s-p_1)^2} = 1s^3 + [11-2]s^2 + [45-22+4]s + [81-90+44-8]$$

or:

$$\frac{F(s)}{(s-p_1)^2} = s^3 + 9s^2 + 27s + 27 = (s+3)^3$$

Applying equation 1-21:

$$\frac{F(s)}{(s-p_2)^3} = A_5(p_2)s^2 + [A_4(p_2)+2p_2A_5(p_2)]s + [A_3(p_2)+2p_2A_4(p_2)$$

$$+3p_2{}^2A_5(p_2)]$$

Evaluating:

$$\frac{F(s)}{(s-p_2)^3} = 1s^2 + [10-6]s + [37-60+27]$$

or:

$$\frac{F(s)}{(s-p_2)^3} = s^2 + 4s + 4 = (s+2)^2$$

<u>Theorem 1-3</u>

Any result which involves an associated polynomial, $A_j(p_i)$, evaluated at a particular root, can always be interpreted as a result involving elementary symmetric functions,

i.e.,

$$A_j(p_i) = (-1)^j \sigma_{n-j} \; : \; j = 1,2,\ldots,n-1$$

where σ_{n-j} denotes the $n-j^{-th}$ elementary symmetric function of the roots of F(s) with p_i absent.

Proof:

Elementary symmetric functions are defined [5] as follows: If f(x) is a polynomial of degree n with complex coefficients, the functions obtained by taking the sum of all the roots of f(x), the sum of all products of pairs of the roots of f(x), the sum of all products of triplets of the roots of f(x) and so on to the product of all the roots of f(x) may be expressed rationally in terms of the coefficients of f(x). These functions of the roots of a polynomial, f(x), are called elementary symmetric functions.

Hence, for a polynomial, F(s), of degree n with roots, $p_1, p_2, \ldots, p_n$:

$$F(s) = (s-p_1)(s-p_2)\ldots(s-p_n) = s^n - \sigma_1 s^{n-1} + \sigma_2 s^{n-2} - \ldots + (-1)^n \sigma_n \tag{1-24}$$

where:

$$\sigma_1 = p_1 + p_2 + \ldots + p_n \tag{1-25a}$$

$$\sigma_2 = p_1p_2 + p_1p_3 + \ldots + p_{n-1}p_n \tag{1-25b}$$

$$\ldots$$

$$\sigma_n = p_1p_2p_3\ldots p_n \tag{1-25c}$$

$\sigma_1, \sigma_2, \ldots, \sigma_n$ are the elementary symmetric function of the roots, $p_1, p_2, \ldots, p_n$.

Rewriting equation 1-17 as:

$$G(s) = \frac{F(s)}{s-p_i} = A_1(p_i) + A_2(p_i)s + \ldots + A_n(p_i)s^{n-1} \quad (1\text{-}26)$$

G(s) is a polynomial of degree n-1. By comparison with equation 1-24, the associated polynomials of F(s), evaluated at a particular root, p_i, can be seen to be elementary symmetric functions of $p_i, p_2, \ldots, p_n$ with the particular root, $p_i = 0$.

Changing the notation to conform with the already established associated polynomial notation:

$$A_{n-1}(p_i) = -(p_1+p_2+\ldots+p_n);\ p_i = 0 \quad (1\text{-}27a)$$

$$A_{n-2}(p_i) = +(p_1p_2+p_1p_3+\ldots+p_{n-1}p_n);\ p_i = 0 \quad (1\text{-}27b)$$

. . .

Expressing equations 1-27 explicitly:

$$A_j(p_i) = (-1)^j\sigma_{n-j} \text{ with: } p_i = 0 \text{ and } j = 1,2,\ldots,n-1 \quad (1\text{-}28)$$

CHAPTER II

Partial Fraction Expansion Using Associated Polynomials

2.1: Introduction

The basic associated polynomial theory developed in the preceding chapter has immediate application in the field of network analysis and network synthesis. Specifically, associated polynomials may be used as an alternate method for performing the Heaviside or partial fraction expansion of a ratio of polynomials.

In network analysis, a method of finding the impulse response, h(t), from the system's function is to perform a partial fraction expansion of the response's Laplace transform, H(s), and then use standard inverse transform tables. Many network synthesis procedures require that network functions be broken up into the sum of a number of terms which can be identified with network elements. In the case of the well known Cauer procedures, this involves partial fraction expansions.

The use of associated polynomials in partial fraction expansions is not the easiest method available when hand calculations are to be used. However, the calculations involved using associated polynomials are very adaptable to the digital computer. Although an actual computer program is not presented here, this chapter presents the algorithm, or the step-by-step procedure for accomplishing a partial fraction expansion of any ratio of polynomials.

Consider the rational function:

$$C(s) = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i} = \frac{N(s)}{D(s)} \tag{2-1}$$

where $a_n = 1$ and $n \geq m$. If equation 2-1 represents the transformed solution of an n^{-th} order differential equation, then the denominator term is the characteristic equation of the system under study. It can be shown[6] that the roots of the characteristic equation can occur in no more complicated form than a pair of complex conjugate roots or as multiple roots. Therefore, for convenience, partial fraction expansions using associated polynomials are developed using three different cases. These are:

Case (1): C(s) contain first-order poles only,

Case (2): C(s) contains a pair of complex poles plus first-order poles,

Case (3): C(s) contains multiple poles plus first order poles.

2.2: Case (1) Partial Fraction Expansion

If the roots of the denominator of equation 2-1 are each real and distinct, then by the fundamental theorem of algebra, the function, C(s), may be rewritten as:

$$C(s) = \frac{N(s)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \tag{2-2}$$

The partial fraction expansion of equation 2-2 is:

$$C(s) = \frac{k_1}{(s-p_1)} + \frac{k_2}{(s-p_2)} + \ldots + \frac{k_n}{(s-p_n)} \tag{2-3}$$

The coefficient, k_i, called the residue of $C(s)$ at $s = p_i$, $i = 1,2,...,n$, is an undetermined coefficient of the expansion and is found by:

$$k_i = \left[\frac{N(s)}{D(s)} (s-p_i)\right]_{s=p_i} \tag{2-4}$$

Equation 2-4 is the classical method of finding the residues of a partial fraction expansion when all the roots are simple and distinct. This calculation is easy to perform by hand calculations but requires several extra steps on a digital computer which will not divide by zero. Associated polynomial theory may be used instead of the above computations by the following method.

Rearrange equation 2-4 as:

$$k_i = \left[\frac{N(s)}{D(s)/(s-p_i)}\right]_{s=p_i} \qquad i = 1,2,...,n \tag{2-5}$$

The denominator of equation 2-5 can be expressed using equation 1-17 as:

$$\frac{D(s)}{s-p_i} = \sum_{j=1}^{n} s^{j-1} A_j(p_i) \tag{2-6}$$

where $A_j(p_i)$ is the set of associated polynomials of $D(s)$ evaluated at $s = p_i$. Hence equation 2-5 can be expressed as:

$$k_i = \frac{N(p_i)}{\sum_{j=1}^{n} p_i^{\,j-1} A_j(p_i)} \qquad i = 1,2,...,n \tag{2-7}$$

<u>Example 2-1</u>

Expand $C(s) = \dfrac{12s^2 + 22s + 6}{s^3 + 3s^2 + 2s}$ by a partial fraction expansion.

Solution:

$$C(s) = \frac{N(s)}{D(s)} = \frac{12s^2+22s+6}{s^3+3s^2+2s} = \frac{k_1}{s} + \frac{k_2}{s+1} + \frac{k_3}{s+2}$$

Let: $p_1 = 0$; $p_2 = -1$; $p_3 = -2$.

Form the associated polynomials of the denominator, D(s):

$$A_3(u) = 1$$

$$A_2(u) = 3 + u$$

$$A_1(u) = 2 + 3u + u^2$$

Applying equation 2-7, the partial fraction coefficients are:

$$k_1 = \frac{N(s=0)}{A_1(0)+0} = \frac{6}{2} = 3$$

$$k_2 = \frac{N(s=-1)}{A_1(-1)+(-1)A_2(-1)+(-1)^2A_3(-1)}$$

where:

$$A_1(-1) = 0;\ A_2(-1) = 2;\ A_3(-1) = 1$$

and:

$$N(s=-1) = 12 - 22 + 6 = -4$$

Hence:

$$k_2 = \frac{-4}{0-2+1} = 4$$

$$k_3 = \frac{N(s=-2)}{A_1(-2)+(-2)A_2(-2)+(-2)^2A_3(-2)}$$

where:

$$A_1(-2) = 0;\ A_2(-2) = 1;\ A_3(-2) = 1$$

and

$$N(s=-2) = 48 - 44 + 6 = 10$$

Hence:

$$k_3 = \frac{10}{0+1(-2)+1(4)} = 5$$

The partial fraction expansion of the function, C(s), is:

$$C(s) = \frac{12s^2+22s+6}{s^3+3s^2+2s} = \frac{3}{s} + \frac{4}{s+1} + \frac{5}{s+2}$$

2.3: Case (2) Partial Fraction Expansion

When the denominator of equation 2-1 contains a pair of complex roots plus first order poles, then the function, C(s), is written:

$$C(s) = \frac{N(s)}{(s-p_1)(s-p_1^*)(s-p_2)\ldots(s-p_{n-1})} \qquad (2\text{-}8)$$

The partial fraction expansion of equation 2-8 is:

$$C(s) = \frac{k_1}{(s-p_1)} + \frac{k_1^*}{(s-p_1^*)} + \frac{k_2}{(s-p_2)} + \ldots + \frac{k_{n-1}}{(s-p_{n-1})} \qquad (2\text{-}9)$$

As shown in reference 6, the coefficients, k_1 and k_1^*, must be complex conjugates and either may be found by the same technique used in case (1) above.

<u>Example 2-2</u>

Expand $C(s) = \dfrac{4}{s(s^2+2s+2)}$ by a partial fraction expansion.

Solution:

$$C(s) = \frac{N(s)}{D(s)} = \frac{4}{s(s^2+2s+2)} = \frac{k_1}{s} + \frac{k_2}{s+1-j} + \frac{k_2^*}{s+1+j}$$

Let $p_1 = 0$; $p_2 = -1+j$; $p_2^* = -1-j$.

Form the associated polynomials of the denominator, D(s):

$$A_3(u) = 1$$

$$A_2(u) = 2 + u$$

$$A_3(u) = 2 + 2u + u^2$$

Applying equation 2-7, the partial fraction coefficients are:

$$k_1 = \frac{N(0)}{A_1(0)+0} = \frac{4}{2} = 2$$

$$k_2 = \frac{N(-1+j)}{A_1(-1+j)+(-1+j)A_2(-1+j)+(-1+j)^2 A_3(-1+j)}$$

Evaluating:

$$A_1(-1+j) = 2 - 2 + 2j - 2j = 0$$

$$A_2(-1+j) = 1 + j$$

$$A_3(-1+j) = 1$$

$$N(-1+j) = 4$$

hence:

$$k_2 = \frac{-2}{1+j} = -1 + j$$

As the coefficient k_2^* must be the conjugate of k_2,

$$\underline{k_2^* = -1-j}$$

The partial fraction expansion of the function, C(s), is:

$$C(s) = \frac{4}{s(s^2+20+2)} = \frac{2}{s} + \frac{-1+j}{s+1-j} + \frac{-1-j}{s+1+j}$$

2.4: Case (3) Partial Fraction Expansion

When the function, C(s), is of the form:

$$C(s) = \frac{N(s)}{(s-p_1)^K(s-p_2)\ldots(s-p_{n-K})} = \frac{N(s)}{D(s)} \qquad (2\text{-}10)$$

The partial fraction expansion is:

$$C(s) = \frac{k_m}{(s-p_1)^K} + \frac{k_{m-1}}{(s-p_1)^{K-1}} + \ldots + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \ldots + \frac{k_{n-K}}{s-p_{n-K}}$$

(2-11)

As shown by equation 2-11 the partial fraction expansion is modified to include as many additional coefficients as the order of the multiple root.

Several methods[6,7] are available for the evaluation of the coefficients of the multiple roots. Computer programs are also available[8,9] that perform partial fraction expansions of expressions containing multiple roots but as shown in reference 7, there are several inherent sources of error that tend to degrade the numerical accuracy as the order of multiplicity of the root increases.

This section introduces a procedure for expanding a multiple root expansion, such as equation 2-10, using a combination of associated polynomial theory and simple algebraic manipulations. The only source of error in the method is that the denominator polynomial, D(s), must be in factored form.

The classical method[10] of determining the partial fraction expansion of the function, C(s) is:

$$C(s) = \sum_{i=1}^{N} \sum_{K=1}^{N_i} \frac{c_{iK}}{(s-p_i)^K} \qquad (2\text{-}12)$$

where the coefficients, c_{iK}, are given by:

$$c_{iK} = \frac{1}{(N_i-K)!} \frac{d^{N_i-K}}{ds^{N_i-K}} \left[(s-p_i)^{N_i} C(s) \right]_{s=p_i} \qquad (2\text{-}13)$$

The interdependence of the successive derivatives in equation 2-13 is a possible cause of errors which cumulate as the multiplicity, N_i, increases. A second method of

determining the coefficients, c_{iK}, using algebraic manipulations only is demonstrated in examples 2-3 and 2-4.

Example 2-3

Given the rational function:

$$C(s) = \frac{1}{s(s+1)^2} = \frac{k_1}{s} + \frac{k_2}{(s+1)^2} + \frac{k_3}{(s+1)} \qquad (2\text{-}14)$$

Using the classical method of determining the coefficients, k_1 and k_2, are found to be:

$$k_1 = s\,C(s)\Big|_{s=0} = 1$$

$$k_2 = (s+1)^2\,C(s)\Big|_{s=-1} = 1/-1 = -1$$

To find the coefficient, k_3, multiply both sides of equation 2-14 by the denominator of k_3, i.e., (s+1). Using the known values of k_1 and k_2, combine the terms that do not go to zero as $s \to -1$. There will be a cancellation of the undeterminate terms after the combination of terms. After the cancellation, let s = -1 and solve for k_3.

$$(s+1)\left[\frac{1}{s(s+1)^2}\right] = (s+1)\frac{k_1}{s} + \frac{(s+1)k_2}{(s+1)^2} + \frac{(s+1)k_3}{(s+1)}$$

Let $s \to -1$ and combine the non-zero terms:

$$k_3 = \frac{1}{s(s+1)} - \frac{k_2}{(s+1)}$$

but $k_2 = -1$, hence:

$$k_3 = \frac{(s+1)}{s(s+1)} \; .$$

Let s = -1, therefore the coefficient, k_3, is: $k_3 = -1$.

The denominator of the unknown coefficient will always be cancelled when the above procedure is used. The resulting partial fraction expansion is:

$$C(s) = \frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{(s+1)}$$

Example 2-4

$$C(s) = \frac{s+2}{(s+1)(s+3)^3} = \frac{k_1}{(s+1)} + \frac{k_2}{(s+3)^3} + \frac{k_3}{(s+3)^2} + \frac{k_4}{(s+3)} \tag{2-15}$$

As before:

$$k_1 = (s+1)C(s) \Big|_{s=-1} = \frac{-1+2}{(-1+3)^3} = \frac{1}{8}$$

and

$$k_2 = (s+3)^3 C(s) \Big|_{s=-3} = \frac{-3+2}{-3+1} = \frac{1}{2}$$

To find k_3, multiply each side of equation 2-15 by $(s+3)^2$:

$$\frac{(s+3)^2(s+2)}{(s+3)^3(s+1)} = \frac{(s+3)^2 k_1}{(s+1)} + \frac{(s+3)^2 k_2}{(s+3)^3} + \frac{(s+3)^2 k_3}{(s+3)^2} + \frac{(s+3)^2 k_4}{(s+3)}$$

Combine all terms that do not go to zero as $s \to -3$.

$$k_3 = \frac{s+2}{(s+3)(s+1)} - \frac{k_2}{s+3}$$

but $k_2 = \frac{1}{2}$, hence:

$$k_3 = \frac{s+2-s/2-1/2}{(s+1)\ (s+3)} = \frac{1/2(s+3)}{(s+1)(s+3)}$$

Therefore:

$$k_3 = \frac{1}{2(s+1)} \Big|_{s=-3} = -\frac{1}{4}$$

The same procedure is used to find the coefficient, k_4, i.e.,

$$\frac{(s+3)\ (s+2)}{(s+3)^3(s+1)} = \frac{(s+3)k_1}{s+1} + \frac{(s+3)k_2}{(s+3)^3} + \frac{(s+3)k_3}{(s+3)^2} + \frac{(s+3)k_4}{(s+3)}$$

Let $s \to -3$ and combine all non-zero terms.

$$k_4 = \frac{s+2}{(s+3)^2(s+1)} - \frac{k_2}{(s+3)^2} - \frac{k_3}{s+3}$$

But $k_2 = \frac{1}{2}$ and $k_3 = -\frac{1}{4}$, hence

$$k_4 = \frac{s+2-s/2-1/2+2s/4+3/4}{(s+1)(s+3)^3} = \frac{(s+3)^2}{4(s+1)(s+3)^2}$$

Let $s = -3$, therefore the coefficient, k_4, is: $k_4 = -\frac{1}{8}$

The partial fraction expansion is:

$$C(s) = \frac{s+2}{(s+1)(s+3)^3} = \frac{1/8}{(s+1)} + \frac{1/2}{(s+3)^3} + \frac{-1/4}{(s+3)^2} + \frac{-1/8}{(s+3)}$$

In the preceding two examples, the algebraic manipulations required in the evaluation of the multiple root coefficients are adaptable to the digital computer. The coefficients that were evaluated in the classical manner may be determined by associated polynomials, specifically by the use of Theorem 1-2 and corollary 1-3. The combined operation is detailed below for two possible cases, i.e., when the repeated roots are of order two and three.

Given the function:

$$C(s) = \frac{\sum_{i=0}^{m} b_i s^i}{(s-p_1)(s-p_2)\dots(s-p_m)^2\dots(s-p_{n-2})} = \frac{N(s)}{D(s)} \qquad (2\text{-}16)$$

The partial fraction expansion of C(s) is:

$$C(s) = \frac{k_{m+1}}{(s-p_m)^2} + \frac{k_m}{(s-p_m)} + \frac{k_1}{(s-p_1)} + \ldots + \frac{k_{n-2}}{(s-p_{n-2})} \quad (2\text{-}17)$$

The coefficients, k_1, k_2, ..., k_{n-2}, are found using the procedure outlined in case (1) above. The multiple order coefficients, k_{m+1} and k_m are found by the following method.

$$k_{m+1} = (s-p_m)^2 C(s)\Big|_{s=p_m} = \frac{N(s)}{D(s)/(s-p_m)^2}\Big|_{s=p_m} \quad (2\text{-}18)$$

The denominator of equation 2-18 can be expressed using equation 1-19 as:

$$\frac{D(s)}{(s-p_m)^2} = \sum_{h=0}^{n-2} s^h \sum_{j=h+2}^{n} p_m^{\,j-(h+2)} A_j(p_m) \quad (2\text{-}19)$$

Evaluating at $s = p_m$, the coefficient k_{m+1} equals:

$$k_{m+1} = \frac{N(p_m)}{\sum_{h=0}^{n-2} p_m^{\,h} \sum_{j=h+2}^{n} p_m^{\,j-(h+2)} A_j(p_m)} \quad (2\text{-}20)$$

In order to find the coefficient, k_m, the algebraic method shown in examples 2-3 and 2-4 must be employed, that is, multiply each side of equation 2-17 by the denominator of k_m.

$$(s-p_m)\frac{N(s)}{D(s)} = \frac{(s-p_m)k_{m+1}}{(s-p_m)^2} + \frac{(s-p_m)k_m}{(s-p_m)} + (s-p_m)\left[\frac{k_1}{(s-p_1)} \ldots + \frac{k_{n-2}}{(s-p_{n-2})}\right] \quad (2\text{-}21a)$$

Let $s \to p_m$ and combine each term that does not go to zero.

$$k_{m+1} = \left. \frac{N(s)-k_{m+1}\ [(s-p_1)(s-p_2)\ldots(s-p_{n-2})]}{(s-p_m)(s-p_1)\ldots(s-p_{n-2})} \right|_{s=p_m} \qquad \text{(2-21b)}$$

The numerator of equation 2-21b will always contain a factorable term, $(s-p_m)$. Since $(s-p_m)$ cancels in the combination of the two terms which individually become indeterminate as $s \to p_m$, the combination is finite. Hence:

$$k_m = \left. \frac{\dfrac{N(s)-k_{m+1}(s-p_1)(s-p_2)\ldots(s-p_{n-2})}{(s-p_m)}}{(s-p_1)(s-p_2)\ldots(s-p_{n-2})} \right|_{s=p_m} \qquad \text{(2-22)}$$

Example 2-5

Expand by partial fractions:

$$C(s) = \frac{1}{s(s+1)^2} = \frac{N(s)}{D(s)} = \frac{k_1}{s} + \frac{k_2}{(s+1)} + \frac{k_3}{(s+1)^2}$$

Solution:

Let: $p_1 = 0;\ p_m = -1.$

Form the associated polynomials of the denominator, $D(s)$:

$$A_3(u) = 1$$

$$A_2(u) = 2 + u$$

$$A_1(u) = 1 + 2u + u^2$$

Applying equation 2-7, the coefficient k_1 is:

$$k_1 = \frac{N(0)}{A_1(0)+0} = 1$$

Applying equation 2-20, the coefficient, $k_{m+1} = k_3$ is:

$$k_3 = \frac{N(-1)}{A_3(-1)(-1)+[A_2(-1)+(-1)A_3(-1)]}$$

$$k_3 = \frac{1}{-1+1-1} = -1$$

Applying equation 2-22, the coefficient, $k_m = k_2$ is:

$$k_2 = \left. \frac{\frac{1-(-1)s}{s+1}}{s} \right|_{s=-1} = -1$$

The partial fraction expansion of C(s) is:

$$C(s) = \frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{(s+1)^2} + \frac{1}{(s+1)}$$

which agrees with example 2-3.

When the expression, C(s), contains a third-order pole such that:

$$C(s) = \frac{\sum_{i=0}^{m} b_i s^i}{(s-p_m)^3 (s-p_1)(s-p_2)\ldots(s-p_{n-3})} = \frac{N(s)}{D(s)} \qquad (2\text{-}23)$$

The expanded form is:

$$C(s) = \frac{N(s)}{D(s)} = \frac{k_{m+2}}{(s-p_m)^3} + \frac{k_{m+1}}{(s-p_m)^2} + \frac{k_m}{(s-p_m)} + \frac{k_1}{(s-p_1)} + \ldots + \frac{k_{n-3}}{(s-p_{n-3})}$$

(2-24)

As above, the coefficients, $k_1, k_2, \ldots, k_{n-3}$, are found using the procedure of case (1). The multiple order coefficients, k_{m+2}, k_{m+1}, and k_m are found by the following:

$$k_{m+2} = (s-p_m)^3 C(s) \Big|_{s=p_m} = \frac{N(s)}{D(s)/(s-p_m)^3} \Big|_{s=p_m} \qquad (2\text{-}25)$$

Using equation 1-21, the denominator of equation 2-25 can be expressed as:

$$\frac{D(s)}{(s-p_m)^3} = \sum_{h=1}^{n-2} s^{h-1} \sum_{j=h+2}^{n} [j-(h+1)] p_m^{\,j-(h+2)} A_j(p_m) \qquad (2\text{-}26)$$

Evaluating as $s = p_m$, the coefficient k_{m+2} is:

$$k_{m+2} = \frac{N(p_m)}{\sum_{h=1}^{n-2} p_m^{h-1} \sum_{j=h+2}^{n} [j-(h+1)] p_m^{j-(h+2)} A_j(p_m)} \qquad (2\text{-}27)$$

The coefficients, k_{m+1} and k_m, are evaluated as in the second-order example above. These coefficients are found to be:

$$k_{m+1} = \left. \frac{\dfrac{N(s)-k_{m+2}(s-p_1)(s-p_2)\ldots(s-p_{n-3})}{(s-p_m)}}{(s-p_1)(s-p_2)\ldots(s-p_{n-3})} \right|_{s=p_m} \qquad (2\text{-}28)$$

and

$$k_m = \left. \frac{\dfrac{N(s)-k_{m+2}(s-p_1)\ldots(s-p_{n-3})-k_{m+1}(s-p_m)\ldots(s-p_{n-3})}{(s-p_m)^2}}{(s-p_1)(s-p_2\ldots(s-p_{n-3})} \right|_{s=p_m} \qquad (2\text{-}29)$$

<u>Example 2-6</u>

Expand by partial fractions:

$$C(s) = \frac{s+2}{(s+1)(s+3)^3} = \frac{N(s)}{D(s)} = \frac{k_1}{s+1} + \frac{k_2}{(s+3)^3} + \frac{k_3}{(s+3)^2} + \frac{k_4}{(s+3)}$$

Solution:

Let: $p_1 = -1$; $p_m = -3$

Form the associated polynomials of the denominator, D(s):

$$D(s) = s^4 + 10s^3 + 36s^2 + 54s + 27$$

$$A_4(u) = 1$$

$$A_3(u) = 10 + u$$

$$A_2(u) = 36 + 10u + u^2$$

$$A_1(u) = 54 + 36u + 10u^2 + u^3$$

Applying equation 2-7, the coefficient, k_1, is:

$$k_1 = \frac{N(-1)}{A_1(-1)+(-1)A_2(-1)+(-1)^2A_3(-1)+(-1)^3A_4(-1)}$$

where: $N(-1) = 1$;

$A_1(-1) = 27$; $A_2(-1) = 27$; $A_3(-1) = 9$; $A_4(-1) = 1$.

Hence:

$$k_1 = \frac{1}{27-27+9-1} = \frac{1}{8}$$

Applying equation 2-27, the coefficient, k_2, is:

$$k_2 = \frac{N(-3)}{(-3)A_4(-3)+[A_3(-3)+2(-3)A_4(-3)]}$$

where:

$$N(-3) = -1;\ A_3(-3) = 7;\ A_4(-3) = 1.$$

Hence:

$$k_2 = \frac{-1}{-3+7-6} = \frac{1}{2}$$

Applying equation 2-28, the coefficient, k_3, is:

$$k_3 = \frac{\dfrac{s+2-1/2(s+1)}{s+3}}{s+1} = \left.\frac{1/2}{s+1}\right|_{s\,=\,-3}$$

$$k_3 = -1/4$$

Applying equation 2-29, the coefficient, k_4, is:

$$k_4 = \frac{\dfrac{s+2-1/2(s+1)+1/4(s+1)(s+3)}{(s+3)^2}}{s+1} = \left.\frac{1}{4(s+1)}\right|_{s\,=\,-3}$$

$$k_4 = -1/8$$

CHAPTER III

Polynomial Sensitivity Theory

3.1: Introduction

In this chapter several simple algebraic operations involving polynomial sensitivity are investigated. Specifically, associated polynomial theory is used to study the relationships between variations of polynomial coefficients and polynomial roots. Both large and incremental variations are considered. As is shown in the next two chapters, the results derived from this section have very useful applications, particularly in the field of sensitivity analysis (Chapter Four) and in self-adaptive control systems (Chapter Five). In order to obtain these results, a basic orthonormality relationship between the set of associated polynomials and the roots of the defining polynomial is defined. In addition to being used in the derivation of sensitivity expressions, the orthonormality relationship can be used to solve a variety of linear problems that involve matrix inversion.

3.2: Coefficient Sensitivity

The effect of variations of a root on the coefficients of a polynomial can be expressed by the following theorem:

THEOREM 3-1

The variations of the coefficients of a polynomial with respect to the variation of a root may be determined using polynomials by the relationship:

$$\frac{\Delta a_{j-1}}{\Delta p_i} = -A_j(p_i) \quad : i,j = 1,2,\ldots,n$$

This result holds for large variations when p_i is a simple root.

Proof:

Let a monic polynomial, F(s), of degree n with n distinct roots be expressed as:

$$F(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0 = \sum_{j=0}^{n} a_j s^j \; ; \; a_n = 1 \tag{3-1}$$

By the fundamental theorem of Algebra, the polynomial may also be written as:

$$F(s) = \prod_{h=1}^{n} (s-p_h) = f(p_1,p_2,\ldots,p_i,\ldots,p_n) \tag{3-2}$$

Let Δp_i denote a change in the root, p_i. The polynomial, $f(p_1,p_2,\ldots,p_i+\Delta p_i,\ldots,p_n)$, may be expressed in a Taylor series expansion about the point $(p_1,p_2,\ldots,p_i,\ldots,p_n)$ by:

$$\begin{aligned} f(p_1,p_2,\ldots p_i+\Delta p_i,\ldots p_n) &= f(p_1,p_2,\ldots,p_i,\ldots,p_n) + \\ &\frac{\partial f(p_1,p_2,\ldots,p_i,\ldots,p_n)}{\partial p_i} \Delta p_i + \\ &1/2 \, \frac{\partial^2 f(p_1,p_2,\ldots,p_i,\ldots,p_n)}{\partial p_i^{\,2}} (\Delta p_i)^2 + \\ &\ldots + \frac{1}{n!} \frac{\partial^n f(p_1,p_2,\ldots,p_i,\ldots,p_n)}{\partial p_i^{\,n}} (\Delta p_i)^n \end{aligned} \tag{3-3}$$

However, from the factored form given in equation 3-2:

$$\frac{\partial f(p_1,p_2,\ldots,p_i,\ldots,p_n)}{\partial p_i} = - \frac{F(s)}{(s-p_i)} \tag{3-4}$$

and:

$$\frac{\partial^n f(p_1, p_2, \ldots, p_i, \ldots, p_n)}{\partial p_i^{\,n}} = 0 \; : \; n \neq 1 \qquad (3\text{-}5)$$

Equation 3-5 applies when p_i is a simple root only. Hence equation 3-3 becomes:

$$f(p_1, p_2, \ldots, p_i + \Delta p_i, \ldots, p_n) = f(p_1, p_2, \ldots, p_n) - \Delta p_i \frac{F(s)}{s - p_i} \qquad (3\text{-}6)$$

and:

$$\Delta F = -\Delta p_i \frac{F(s)}{s - p_i} \qquad (3\text{-}7)$$

where:

$$\Delta F = f(p_1, p_2, \ldots, p_i + \Delta p_i, \ldots, p_n) - f(p_1, p_2, \ldots, p_n) \qquad (3\text{-}8)$$

The expression ΔF in equation 3-8 may be expressed in terms of the changes in the coefficient, Δa_k, of the original polynomial, F(s). Note that:

$$f(p_1, p_2, \ldots, p_i + \Delta p_i, \ldots, p_n) = (s - p_1) \ldots (s - p_i + \Delta p_i) \ldots (s - p_n) \qquad (3\text{-}9)$$

Substituting equations 3-2 and 3-9 into equation 3-8 and subtracting the coefficients of equal power of s reveals that:

$$\Delta F = \Delta a_0 + \Delta a_1 s + \ldots + \Delta a_{n-1} s^{n-1} = \sum_{j=1}^{n} \Delta a_{j-1} s^{j-1} \qquad (3\text{-}10)$$

Using Theorem 1-2, i.e.,;

$$\frac{F(s)}{s - p_i} = \sum_{j=1}^{n} A_j (p_i) s^{j-1} \qquad (1\text{-}17)$$

Equation 3-7 is rewritten as:

$$\Delta F = -\Delta p_i \sum_{j=1}^{n} A_j(p_i)s^{j-1} \tag{3-11}$$

Equating equations 3-10 and 3-11:

$$\Delta F = \sum_{j=1}^{n} \Delta a_{j-1}s^{j-1} = -\Delta p_i \sum_{j=1}^{n} A_j(p_i)s^{j-1} \tag{3-12}$$

Equating the coefficients of equation 3-12, it follows that:

$$\Delta a_{j-1} = -\Delta p_i A_j(p_i) \quad : i,j = 1,2,\ldots,n \tag{3-13}$$

Thus $-A_j(p_i)$ may be characterized as the change in the coefficient a_{j-1} per unit change in the root, p_i. Note that equation 3-13 holds in the large, i.e., for arbitrary changes in Δp_i, when the root is simple. Similar precise equations may be derived for repeated roots. Equation 3-13 is useful for determining the change in the coefficients of a polynomial when there is a change in a simple root. Equation 3-13 cannot be used to determine the change in root locations caused by changes in any coefficient of a polynomial since a change in general can be expected to change the location of several roots. Thus the inverse of equation 3-13 is not meaningful.

Corollary 3-1

Taking the limit of equation 3-13 as the change in the root, p_i, goes to zero, then:

$$\frac{\partial a_{j-1}}{\partial p_i} = -A_j(p_i) \quad : i,j = 1,2,\ldots,n \tag{3-14}$$

Example 3-1

Given the polynomial:

$$F(s) = s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3)$$

The associated polynomials are:

$$A_3(u) = 1$$

$$A_2(u) = 6 + u$$

$$A_1(u) = 11 + 6u + u^2$$

Let the roots of F(s) be defined as:

$$p_1 = -1;\ p_2 = -2;\ p_3 = -3$$

Assume p_1 is decreased by 8 to a new value $p_1 + \Delta p_1 = -9$.

Hence:

$$\Delta p_1 = -8$$

The original polynomial is changed to:

$$F_1(s) = s^3 + 14s^2 + 51s + 54 = (s+9)(s+2)(s+3)$$

By inspection, the change in the coefficients are seen to be:

$$\Delta a_0 = 48;\ \Delta a_1 = 40;\ \Delta a_2 = 8;\ \Delta a_3 = 0$$

In order to verify theorem 3-1 numerically, the changes in the coefficients can also be found by use of equation 3-13.

For i = 1:

Let j = 3, or:

$$\Delta a_2 = -A_3(p_1)\Delta p_1$$

numerically:

$$\Delta a_2 = -(1)(-8) = 8;$$

which agrees.

Let j = 2, or:

$$\Delta a_1 = -A_2(p_1)\Delta p_1$$

numerically:

$$\Delta a_1 = -[6+(-1)](-8) = 40;$$

which agrees.

Let j = 1, or:

$$\Delta a_0 = -A_1(p_1)\Delta p_1$$

numerically:

$$\Delta a_0 = -[11+6(-1)+(-1)^2](-8) = 48;$$

which agrees.

3.3: The Basic Orthonormality Relation

In order to derive the inverse of Theorem 3-1, that is, the variation of a root of a polynomial with the variation of its coefficients, the orthonormality relationship of associated polynomials must first be derived. Prior to this derivation, Theorem 3-2 is introduced.

<u>THEOREM 3-2</u>

The derivative of a polynomial may also be found from its associated polynomials by:

$$\frac{dF(u)}{du} = F'(u) = \sum_{j=1}^{n} u^{j-1}A_j(u)$$

Proof:

It was shown in the proof of Theorem 1-2 that:

$$\frac{F(s)-F(u)}{s-u} = \sum_{j=1}^{n} s^{j-1}A_j(u) \qquad (1\text{-}14)$$

Let $s \to u$, then

$$\lim_{s\to u} \frac{F(s)-F(u)}{s-u} = \lim_{s\to u} \sum_{j=1}^{n} s^{j-1}A_j(u) \qquad (3\text{-}15)$$

or

$$F'(u) = \sum_{j=1}^{n} u^{j-1} A_j(u) \qquad (3\text{-}16)$$

Example 3-2

Given the polynomial:

$$F(s) = s^4 + 10s^3 + 35s^2 + 50s + 24 = (s+1)(s+2)(s+3)(s+4)$$

The associated polynomials are:

$$A_4(u) = 1$$

$$A_3(u) = 10 + u$$

$$A_2(u) = 35 + 10u + u^2$$

$$A_1(u) = 50 + 35u + 10u^2 + u^3$$

The derivative of the polynomial, F(s), is expressable in two ways:

(1): Classical Method:

$$\frac{dF(s)}{ds} = 4s^3 + 30s^2 + 70s + 50$$

(2): Associated polynomial method:

Applying Theorem 3-2;

$$\begin{aligned}\frac{dF(s)}{ds} &= [A_1(u)+A_2(u)u+A_3(u)u^2+A_4(u)u^3]_{u=s} \\ &= 50 + 35u + 10u^2 + u^3 \\ &\qquad + 35u + 10u^2 + u^3 \\ &\qquad\qquad + 10u^2 + u^3 \\ &\qquad\qquad\qquad + u^3 \\ \hline \frac{dF(s)}{ds} &= 50 + 70u + 30u^2 + 4u^3 \Big|_{u=s}\end{aligned}$$

THEOREM 3-3

The basic orthonormality relationship between the set of associated polynomials and the roots of the defining polynomial is:

$$\sum_{i=1}^{n} \frac{A_j(p_i)p_i^{k-1}}{F'(p_i)} = \delta_{kj} \qquad j,k = 1,2,\ldots,n$$

where δ_{kj} is the Kronecker delta function which vanishes except when the indices are equal.

Proof:

The derivative of a polynomial is expressable using Theorem 3-2 as:

$$F'(u) = \sum_{j=1}^{n} u^{j-1}A_j(u) \qquad (3\text{-}16)$$

If p_i and p_k are roots of the defining polynomial, $F(s)$, then it can be shown that, if p_i is a simple root of $F(s)$:

$$0 = \sum_{j=1}^{n} p_k^{j-1}A_j(p_i) \qquad (3\text{-}17)$$

From equations 3-16 and 3-17, it follows that:

$$\frac{\sum_{j=1}^{n} p_k^{j-1}A_j(p_i)}{\sum_{j=1}^{n} p_i^{j-1}A_j(p_i)} = \begin{cases} 0 \text{ when } i \neq k \\ 1 \text{ when } i = k \end{cases} \qquad (3\text{-}18)$$

Equation 3-18 may be rewritten:

$$\sum_{j=1}^{n} \frac{A_j(p_i)p_k^{j-1}}{F'(p_i)} = \delta_{ik} \qquad i,k = 1,2,\ldots,n \qquad (3\text{-}19)$$

where δ_{ik} is the Kronecker delta function.

Let $[\alpha]$ and $[\beta]$ denote the matrices who elements are:

$$\alpha_{ij} = \frac{A_j(p_i)}{F'(p_i)} \quad j = 1,2,\ldots,n;\ i = 1,2,\ldots,n \tag{3-20a}$$

and

$$\beta_{jk} = p_k^{\,j-1} \quad j = 1,2,\ldots,n;\ k = 1,2,\ldots,n \tag{3-20b}$$

If I denotes the identity matrix, then equation 3-19 may be written:

$$[\alpha][\beta] = \sum_{j=1}^{n} \alpha_{ij}\beta_{jk} = \delta_{ik} = I \tag{3-21}$$

Since β_{jk} is a Vandermonde matrix, it is non-singular and:

$$[\beta][\alpha] = \sum_{i=1}^{n} \beta_{ki}\alpha_{ij} = I \tag{3-22}$$

Hence:

$$\sum_{i=1}^{n} \frac{A_j(p_i)p_i^{\,k-1}}{F'(p_i)} = \delta_{kj} \quad j,k = 1,2,\ldots,n \tag{3-23}$$

Theorem 3-3 may also be verified by the following example:

Example 3-3

$$F(s) = s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3)$$

Let:

$$p_1 = -1;\ p_2 = -2;\ p_3 = -3.$$

The associated polynomials of the given polynomial are:

$$A_3(u) = 1$$

$$A_2(u) = 6 + u$$

$$A_1(u) = 11 + 6u + u^2$$

The derivatives of the polynomial evaluated at the three roots are found using equation 3-16 to be:

$$F'(p_i) = A_1(p_i) + A_2(p_i)p_i + A_3(P_i)p_i^2$$

$$F'(p_i) = 6 + 5(-1) + 1 = 2$$

$$F'(p_2) = 3 + 4(-2) + 4 = -1$$

$$F'(p_3) = 2 + 3(-3) + 9 = 2$$

Forming the $[\alpha]$ and $[\beta]$ matrices according to equation 3-20:

$$[\alpha_{ij}] = \begin{bmatrix} \frac{A_1(p_1)}{F'(p_1)} & \frac{A_2(p_1)}{F'(p_1)} & \frac{A_3(p_1)}{F'(p_1)} \\ \frac{A_1(p_2)}{F'(p_2)} & \frac{A_2(p_2)}{F'(p_2)} & \frac{A_3(p_2)}{F'(p_2)} \\ \frac{A_1(p_3)}{F'(p_3)} & \frac{A_2(p_3)}{F'(p_3)} & \frac{A_3(p_3)}{F'(p_3)} \end{bmatrix} = \begin{bmatrix} 6/2 & 5/2 & 1/2 \\ 3/-1 & 4/-1 & 1/-1 \\ 2/2 & 3/2 & 1/2 \end{bmatrix}$$

Or

$$[\alpha_{ij}] = \begin{bmatrix} 3 & 5/2 & 1/2 \\ -3 & -4 & -1 \\ 1 & 3/2 & 1/2 \end{bmatrix}$$

$$[\beta_{jk}] = \begin{bmatrix} p_1^0 & p_2^0 & p_3^0 \\ p_1^1 & p_2^1 & p_3^1 \\ p_1^2 & p_2^2 & p_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

It can be shown that:

$$[\alpha_{ij}][\beta_{jk}] = I \qquad (3\text{-}21)$$

Likewise:

$$[\beta_{jk}][\alpha_{ij}] = I \qquad (3\text{-}22)$$

Expanding the terms in equation 3-22:

$$\begin{bmatrix} p_1^{\,0} & p_2^{\,0} & p_3^{\,0} \\ p_1^{\,1} & p_2^{\,1} & p_3^{\,1} \\ p_1^{\,2} & p_2^{\,2} & p_3^{\,2} \end{bmatrix} \begin{bmatrix} \frac{A_1(p_1)}{F'(p_1)} & \frac{A_2(p_1)}{F'(p_1)} & \frac{A_3(p_1)}{F'(p_1)} \\ \frac{A_1(p_2)}{F'(p_2)} & \frac{A_2(p_2)}{F'(p_2)} & \frac{A_3(p_2)}{F'(p_2)} \\ \frac{A_1(p_3)}{F'(p_3)} & \frac{A_2(p_3)}{F'(p_3)} & \frac{A_3(p_3)}{F'(p_3)} \end{bmatrix} = I$$

Hence:

$$p_1^{\,0}\,\frac{A_1(p_1)}{F'(p_1)} + p_2^{\,0}\,\frac{A_1(p_2)}{F'(p_2)} + p_3^{\,0}\,\frac{A_1(p_3)}{F'(p_3)} = 1 \qquad (3\text{-}24a)$$

Or in matrix notation,

$$[\beta_{j=1,k}][\alpha_{i,j=1}] = 1 \qquad (3\text{-}24b)$$

likewise:

$$[\beta_{j=2,k}][\alpha_{i,j=2}] = 1 \qquad (3\text{-}24c)$$

and:

$$[\beta_{j=3,k}][\alpha_{i,j=3}] = 1 \qquad (3\text{-}24d)$$

In addition, the following relationships also hold:

$$[\beta_{j=1,k}][\alpha_{i,j=2}] = 0 \qquad (3\text{-}25a)$$

$$[\beta_{j=1,k}][\alpha_{i,j=3}] = 0 \qquad (3\text{-}25b)$$

$$\vdots$$

$$[\beta_{j=3,k}][\alpha_{i,j=2}] = 0 \qquad (3\text{-}25c)$$

The above matrix products, equations 3-24 and 3-25 can be expressed explicitly as stated in Theorem 3-3, i.e.,

$$\sum_{i=1}^{n} \frac{A_j(p_i)\,p_i^{\,k-1}}{f'(p_i)} = \delta_{kj} \qquad (3\text{-}23)$$

The orthonormality relation given by equation 3-23 is a basic result which permits one to solve a variety of linear problems. As has been shown in references 1 and 11, this orthonormality relation permits the theoretical investigation of inverse matrix problems, such as finding the inverse to the Vandermonde Matrix in a form suited for numerical calculations.

3.4: Root Sensitivity

The effect of variations of the coefficients of a polynomial on any particular root is expressable by the following theorem:

THEOREM 3-4

The variation of the roots of a polynomial with respect to the variations of a coefficient of the defining polynomial is determined by the relationship:

$$\frac{\partial p_i}{\partial a_j} = -\frac{p_i^{\,j}}{F'(p_i)} \quad : \quad \begin{array}{l} j = 0,1,\ldots,n-1 \\ i = 1,2,\ldots,n \end{array}$$

Proof:

Corollary 3-1 established that:

$$\frac{\partial a_{j-1}}{\partial p_i} = -A_j(p_i) \quad : \; i,j = 1,2,\ldots,n$$

In addition the orthonormality relation of Theorem 3-3 is:

$$\sum_{i=1}^{n} \frac{A_j(p_i)\,p_i^{\,k-1}}{F'(p_i)} = \delta_{kj} \qquad (3\text{-}23)$$

Let $[\gamma]$ and $[\eta]$ denote the matrices whose elements are:

$$\gamma_{ji} = A_j(p_i) \quad i = 1,2,\ldots,n;\; j = 1,2,\ldots,n \qquad (3\text{-}26a)$$

and

$$\eta_{ik} = \frac{p_i^{\,k-1}}{F'(p_i)} \qquad i = 1,2,\ldots,n;\; k = 1,2,\ldots,n \tag{3-26b}$$

Using the matrix notation introduced above, equation 3-23 can be rewritten:

$$[\gamma][\eta] = I \tag{3-27}$$

Hence, the matrix, γ, (whose elements are $A_j(p_i)$) is the inverse of the matrix, η, (whose elements are $p_i^{\,k-1}/F'(p_i)$) given $i,j,k = 1,2,\ldots,n$. In a similar manner, it has been shown[1] that the matrix with elements $\partial p_i/\partial a_{j-1}$ is the inverse of the matrix whose elements are $\partial a_{j-1}/\partial p_i$, that is, if:

$$[A] = \frac{\partial p_i}{\partial a_{j-1}} \tag{3-28a}$$

and

$$[B] = \frac{\partial a_{k-1}}{\partial p_i} \tag{3-28b}$$

Then:

$$[A][B] = I \tag{3-28c}$$

but from corollary 3-1:

$$[B] = -[\gamma] \tag{3-28d}$$

Hence, from equations 3-27 and 3-28d:

$$[A] = -[\eta] \tag{3-28e}$$

Writing out the terms of the matrices, as given by equations 3-28a and 3-26b yields:

$$\begin{bmatrix} \frac{\partial p_1}{\partial a_0} & \frac{\partial p_1}{\partial a_1} & \cdots & \frac{\partial p_1}{\partial a_{n-1}} \\ \frac{\partial p_2}{\partial a_0} & & & \vdots \\ \vdots & & & \\ \frac{\partial p_n}{\partial a_0} & \cdots & & \frac{\partial p_n}{\partial a_{n-1}} \end{bmatrix} = - \begin{bmatrix} \frac{p_1^{\,0}}{F'(p_1)} & \frac{p_1^{\,1}}{F'(p_1)} & \cdots & \frac{p_1^{\,n-1}}{F'(p_1)} \\ \frac{p_2^{\,0}}{F'(p_2)} & & & \vdots \\ \vdots & & & \\ \frac{p_n^{\,0}}{F'(p_n)} & \cdots & & \frac{p_n^{\,n-1}}{F'(p_n)} \end{bmatrix} \qquad (3\text{-}29)$$

Thus in general:

$$\frac{\partial p_i}{\partial a_j} = -\frac{p_i^{\,j}}{F'(p_i)} \qquad \begin{array}{l} i = 1,2,\ldots,n \\ j = 0,1,\ldots,n-1 \end{array} \qquad (3\text{-}30)$$

<u>Corollary 3-2</u>

The total change in the value of any root of an n^{-th} order polynomial due to an incremental change in any or all coefficients of the polynomial is found by:

$$dp_i = \frac{\partial p_i}{\partial a_0}\, da_0 + \frac{\partial p_i}{\partial a_1}\, da_1 + \ldots + \frac{\partial p_i}{\partial a_{n-1}}\, da_{n-1}$$

where:

$$\frac{\partial p_i}{\partial a_m} = -\frac{p_i^{\,m}}{F'(p_i)} \qquad (3\text{-}30)$$

Substituting and factoring:

$$dp_i = \frac{da_0 + p_i da_1 + \ldots + p_i^{\,n-1} da_{n-1}}{-F'(p_i)} \qquad (3\text{-}31)$$

3.5: Root Solving Using Associated Polynomials

Equation 3-31 of Corollary 3-2 may be used in a root solving process if a polynomial with known simple roots can be found near the unknown polynomial. Since equation 3-31 deals with differentials, the process is limited to

polynomials whose roots are close in value to the known polynomial, i.e., small variations in the coefficients between the two. The process is summarized as:

(1): Find a polynomial of equal order with known, nonrepeated, roots, such that the difference in the coefficient values are "small". The accuracy of this root solving process is determined by the value of the variations in the coefficients.

(2): Using the roots of the known polynomial and the values of the change between the coefficients of the known and unknown polynomials, apply equation 3-31.

(3): The change in the known roots determines the values of the roots of the unknown polynomial.

The following examples demonstrate how this root solving process may be employed.

Example 3-4

The roots of a quadratic polynomial are easily determined by the quadratic formula. However, such analytical expressions, in general, do not exist for higher order polynomials. This example will use a quadratic polynomial as an illustration of the root solving method available with associated polynomials.

Given the quadratic polynomial:

$$G(s) = s^2 + 4.7s + 5.3$$

In order to find the roots of G(s), note that the polynomial, F(s), is:

$$F(s) = s^2 + 5s + 6 = (s+2)(s+3)$$

The variations in the coefficients of F(s) from the given polynomial, G(s), are small enough such that the process may be employed with negligible errors.

Let the roots of F(s) be defined as:

$$p_1 = -2; \; p_2 = -3 \; .$$

Evaluating the denominator of equation 3-31:

$$F'(s) = 2s + 5$$

Therefore:

$$-F'(p_1) = -1; \; -F'(p_2) = +1$$

The variations of the coefficients are identified as:

$$da_0 = -.7; \; da_1 = -.3; \; da_2 = 0.$$

The change in the root, p_1, of F(s) is:

$$dp_1 = \frac{-.7+(-2)(-.3)}{-1} = .1$$

Hence: p_1 of G(s) is $-2 + .1 = -1.9$.

The change in the root, p_2, of F(s) is:

$$dp_2 = \frac{-.7+(-3)(-.3)}{+1} = .2$$

Hence: p_2 of G(s) is $-3 + .2 = -2.8$.

The factors of the polynomial, G(s), are:

$$(s+1.9)(s+2.8)$$

As a check on the above results:

$$(s+1.9)(s+2.8) = s^2 + 4.7s + 5.32$$

<u>Example 3-5</u>

Given the third order polynomial:

$$G(s) = s^3 + 5.85s^2 + 10.44s + 5.39$$

To find the roots of G(s), note that the polynomial, F(s), is:

$$F(s) = s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3)$$

Let the roots of F(s) be defined as:

$$p_1 = -1;\ p_2 = -2;\ p_3 = -3.$$

The denominator of equation 3-31 is:

$$F'(s) = 3s^2 + 12s + 11$$

Therefore:

$$F'(p_1) = 3 - 12 + 11 = 2$$

$$F'(p_2) = 12 - 24 + 11 = -1$$

$$F'(p_3) = 27 - 36 + 11 = +2$$

The variations of the coefficients, [a_i of G(s) - a_i of F(s)], are:

$$da_0 = -.61;\ da_1 = -.56;\ da_2 = -.15$$

The change in the root, p_i, of F(s) is:

$$dp_1 = \frac{da_0 + p_1 da_1 + p_1^2 da_2}{-F'(p_1)}$$

$$dp_1 = \frac{(-.61)+(-1)(-.56)+(-1)^2(-.15)}{-2} = .1$$

Hence: p_1 of G(s) is -1 + .1 = -.9.

The change in the root, p_2, of F(s) is:

$$dp_2 = \frac{da_1 + p_2 da_1 + p_2^2 da_2}{-F'(p_2)}$$

$$dp_2 = \frac{(-.61)+(-2)(-.56)+(-2)^2(-.15)}{+1} = -.09$$

Hence: p_2 of G(s) is -2 - .09 = -2.09.

The change in the root, p_3, of F'(s) is:

$$dp_3 = \frac{da_0 + p_3 da_1 + p_3^2 da_2}{-F'(p_3)}$$

$$dp_3 = \frac{(-.61)+(-3)(-.56)+(-3)^2(-.15)}{-2} = .14$$

Hence: p_3 of G(s) is -3 + .14 = -2.86.

The factors of the polynomial, G(s), are:

$$G(s) = (s+.9)(s+2.09)(s+2.86)$$

As a check on the above results:

$$(s+.9)(s+2.09)(s+2.86) = s^3 + 5.85s^2 + 10.43s + 5.38$$

CHAPTER IV

Sensitivity Coefficients

4.1: Introduction

In the analysis of a system whose motion can be described by a set of differential equations, in addition to obtaining solutions as functions of independent variables, the engineer would like a knowledge of the variations of the solutions with respect to the parameters of the system. Since it is impossible to realize a physical system that is identical to a mathematical model, the influence which parameter variations in the physical system have on the system's behavior is an important condition for reliable design.

Variations in a circuit element will cause a displacement of the poles and zeros of driving-point and transfer functions. As a direct consequence, these variations and resulting displacements affect the network response. One method of approaching this problem is to plot the displaced poles and zeros and then determine the new response. As this method involves considerable work, it has been shown[12] that the work involved may be reduced by testing the driving-point or transfer function to determine incremental changes in a particular parameter. This method is called "Sensitivity Analysis".

Tomovic introduced[13] sensitivity coefficients for error and sensitivity analysis of dynamic systems. He has shown that determination of the influence of parameter tolerances on the behavior is based on a knowledge of these

coefficients. They show the effects of incremental parameter variations on the system's behavior by giving the variation of the system response for any given input.

Systems which contain parameters that undergo large variations are studied in the next chapter. This chapter expresses Tomovic' sensitivity coefficients analytically in terms of associated polynomials. Tomovic' work demonstrated how these coefficients may be obtained from a computer model and analytic expressions for them have not been available. Applications of sensitivity coefficients for error and stability analysis has been amply covered in reference 13 and is not covered here.

4-2: Theory of Sensitivity Coefficients

Any dynamic system may be represented by an input, an output and a transfer function relating the two as shown in figure 4-1.

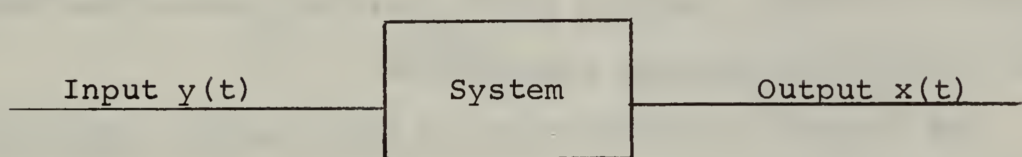

Figure 4-1: Dynamic System Basic Block Diagram

If the Laplace transformation of the input signal, Y(s), is equal to one, then the response of the system is dependent on n parameters and is defined by:

$$X(s,a_0,a_1,\ldots,a_n) = \frac{b_0+b_1s+\ldots+b_ns^n}{a_0+a_1s+\ldots+a_ns^n} \qquad (4\text{-}1)$$

Tomovic introduced the complex sensitivity coefficient, $V_i(s)$, which can be defined with respect to the relative variation of the parameter a_i, as:

$$V_i(s) = \frac{\partial X(s,a_0,a_1,\ldots,a_n)}{\partial \ln a_i} \tag{4-2}$$

It follows from the Laplace integral's property of differentiability with respect to the parameters that the real sensitivity coefficient, $v_i(t)$, can be obtained by the inverse transformation:

$$v_i(t) = \frac{\partial X(t,a_0,a_1,\ldots,a_n)}{\partial \ln a_i} \tag{4-3}$$

In his text, Tomovic continues from this definition to a method of determining the sensitivity coefficients by modeling on an analog computer. This section uses associated polynomial theory for the determination of the sensitivity coefficients in terms of the coefficients of the system's transfer function which therefore makes them amenable to digital computer computation.

The Laplace transformation of the response, equation 4-1, may be written as:

$$X(s,a_i) = \frac{\sum_{j=0}^{n} b_j s^j}{\sum_{i=0}^{n} a_i s^i} = \frac{\left(\frac{1}{an}\right) \sum_{j=0}^{n} b_j s^j}{(s+p_1)(s+p_2)\ldots(s+p_n)} \tag{4-4}$$

where the denominator has been factored. Expanding equation 4-4 into its partial fractions:

$$X(s,a_i) = \frac{k_1}{(s+p_1)} + \frac{k_2}{(s+p_2)} + \ldots + \frac{k_n}{(s+p_n)} \tag{4-5}$$

From equation 4-5, two operations may be noted:

$$(1): \qquad \frac{\partial X(s,a_i)}{\partial p_i} = -\frac{k_i}{(s+p_i)^2} \tag{4-6}$$

$$(2): \qquad \frac{\partial X(s,a_i)}{\partial k_i} = \frac{1}{(s+p_i)} \tag{4-7}$$

The response is a function of the parameters a_i and the residues, k_i.

The complex sensitivity coefficient, equation 4-2, may be written as:

$$V_i(s) = a_i \frac{\partial X(s,a_i)}{\partial a_i} \tag{4-8a}$$

$$= a_i \sum_{j=1}^{n} \left[\frac{\partial X(s,a_i)}{\partial p_j} \frac{\partial p_j}{\partial a_i} + \frac{\partial X(s,a_i)}{\partial k_j} \frac{\partial k_j}{\partial a_i} \right] \tag{4-8b}$$

From associated polynomial theory, the variation of a particular root with the variation of a coefficient is expressable as:

$$\frac{\partial p_i}{\partial a_j} = -\frac{p_i^{\,j}}{F'(p_i)} \qquad \begin{array}{l} i = 1,2,\ldots,n \\ j = 0,1,\ldots,n \end{array} \tag{4-9}$$

For notational purposes, let:

$$\frac{\partial p_1}{\partial a_i} = Q_1; \quad \frac{\partial p_2}{\partial a_i} = Q_2; \quad \ldots; \quad \frac{\partial p_n}{\partial a_i} = Q_n \tag{4-10}$$

Using equations 4-10, 4-6 and 4-7, the complex sensitivity coefficients, equation 4-8 may be simplified to:

$$V_i(s) = a_i \sum_{j=1}^{n} \left[\frac{-k_j}{(s+p_j)^2} Q_j + \frac{1}{(s+p_j)} \frac{\partial k_j}{\partial a_i} \right] \tag{4-11}$$

Equation 4-11 may be further simplified by noting:

$$\frac{\partial k_j}{\partial a_i} = \frac{\partial k_j}{\partial p_j} \frac{\partial p_j}{\partial a_i} \; ; \; j = 1,2,\ldots,n \tag{4-12a}$$

or using equation 4-10,

$$\frac{\partial k_j}{\partial a_i} = \frac{\partial k_j}{\partial p_j} Q_j \; ; \; j = 1,2,\ldots,n \tag{4-12b}$$

The residue, k_j, in equation 4-12b may be expressed in terms of the original equation for the Laplace transformation of the response. From that equation, for notational purposes let:

$$N(s) = \sum_{j=0}^{n} b_j s^j \tag{4-13a}$$

$$D(s) = \sum_{j=0}^{n} a_j s^j \tag{4-13b}$$

Form the set of associated polynomials of the denominator D(s). From associated polynomial theory, the residue k_j is expressable as:

$$k_j = \frac{N(p_j)}{\sum_{m=1}^{n} (p_j)^{m-1} A_m(p_j)} \qquad j = 1,2,\ldots,n \tag{4-14}$$

so that:

$$\frac{\partial k_j}{\partial p_j} = \frac{\partial}{\partial p_j} \left\{ \frac{N(p_j)}{\sum_{m=1}^{n} (p_j)^{m-1} A_m(p_j)} \right\} \tag{4-15}$$

Using equations 4-12 and 4-15, the complex sensitivity coefficient equation 4-11 may be written as:

$$V_i(s) = a_i \left[\frac{Q_1}{(s+p_1)} \left(\frac{-k_1}{s+p_1} + \frac{\partial}{\partial p_1} \left\{ \frac{N(p_1)}{\sum_{m=1}^{n} p_1^{m-1} A_m(p_1)} \right\} \right) + \right.$$

$$\frac{Q_2}{(s+P_2)} \left(\frac{-k_2}{s+p_2} + \frac{\partial}{\partial p_2} \left\{ \frac{N(p_2)}{\sum_{m=1}^{n} p_2^{m-1} A_m(p_2)} \right\} \right) + \cdots$$

$$\left. + \frac{Q_n}{(s+p_n)} \left(\frac{-k_n}{s+p_n} + \frac{\partial}{\partial p_n} \left\{ \frac{N(p_n)}{\sum_{m=1}^{n} p_n^{m-1} A_m(p_n)} \right\} \right) \right] \qquad (4\text{-}16a)$$

Or

$$V_i(s) = a_i \sum_{j=1}^{n} \frac{Q_j}{(s+p_j)} \left[\frac{-k_j}{s+p_j} + \frac{\partial}{\partial p_j} \frac{N(p_j)}{\sum_{m=1}^{n} p_j^{m-1} A_m(p_j)} \right] \qquad (4\text{-}16b)$$

This expression is cumbersome but does give an analytic expression for the transform of a sensitivity function.

CHAPTER V

A Two Parameter Adaptive System Using Associated Polynomials

5.1: Introduction

A feedback control system can be represented by the basic diagram as shown in figure 5-1, or by some modification or extension to it. The plant, control and feedback systems are composed of real physical components. For purposes of analysis and design the physical system is usually modeled mathematically in order that the Laplace Transforms may be employed.

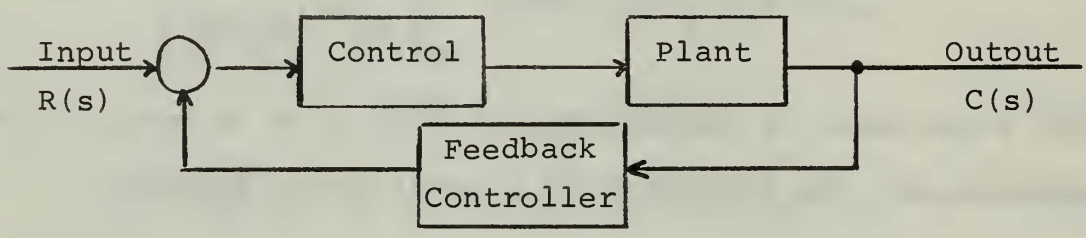

Figure 5-1. Feedback Control System

The characteristic equation of the system is expressable as:

$$1 + HG(s) = 0$$

where:

H(s) is the feedback transfer function

G(s) is the direct or forward transfer function composed of the individual transfer functions of both the control system and the plant.

HG(s) is the open-loop transfer function.

The characteristic equation determines the general form of the system's natural response and its roots give the time

constants of the transient exponential decay factors. Stability of the overall system is found by factoring the characteristic equation. The characteristic equation has real coefficients which are algebraic combinations of physical parameters, which are real numbers.

Basic feedback control theory usually assumes time-invarient elements for purposes of design and analysis. However parameters of the system may be subject to variations due to a changing environment or similar phenomena. If there is a constant loop gain or a fixed compensation scheme and a system parameter were to vary, the closed loop performance could be seriously deteriorated and even system instability occur. Examples of such a changing parameter may be found in different operations such as the change in the aerodynamic parameters of an aircraft as it changes altitudes or the effect of gravity on system parameters in the Shipboard Inertial Navigation System, (SINS), of a deep diving submarine.

If parameter variations are large and slow compared with the normal system's response times, it is possible to design for the capability of continuously measuring the variations and then changing compensation so that the system's performance criteria are always satisfied. Such systems which automatically exhibit the characteristics of adapting to variations in system parameters, are called Self-Adaptive control systems.

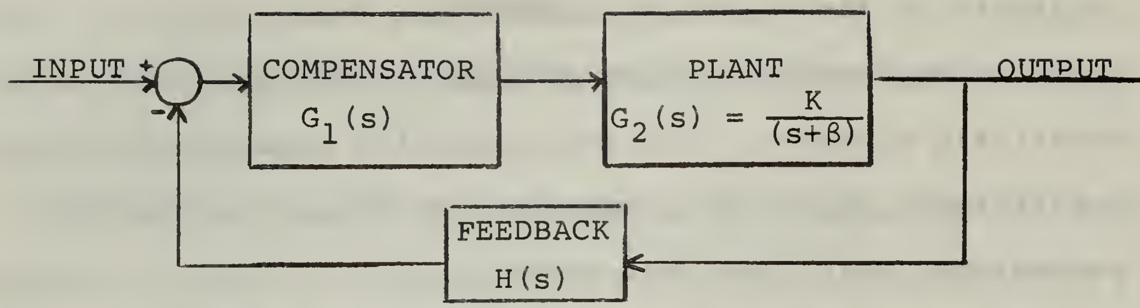

Figure 5-2. Control System with Variable Parameter β

The parameter β of the plant shown in figure 5-2, is known to vary slowly with time or environment.* This parameter appears algebraically in the coefficients of the system's characteristic equation. Since the dominant root values are determined by the coefficients of the defining equation, the system's dominant pole location is a function of the value of the variable parameter β. As the value of β varies so also will the location of the dominant root or roots of the characteristic equation hence the system's response to an input and decay factors will change. If the new value of β moves the location of the dominant root or roots into the right-hand s-plane, the system becomes unstable.

A block diagram of an adaptive control system is shown in figure 5-3. As before the plant has a parameter β which is known to vary with time or environment. The "identification and adapter" block continuously measures the input m(t),

*In the analysis which follows, the assumption of quasi-stationary responses is made in which parameters are assumed to be varying slowly such that the response is approximately given by the system's response with the fixed parameters replaced by a time varying function.(23)

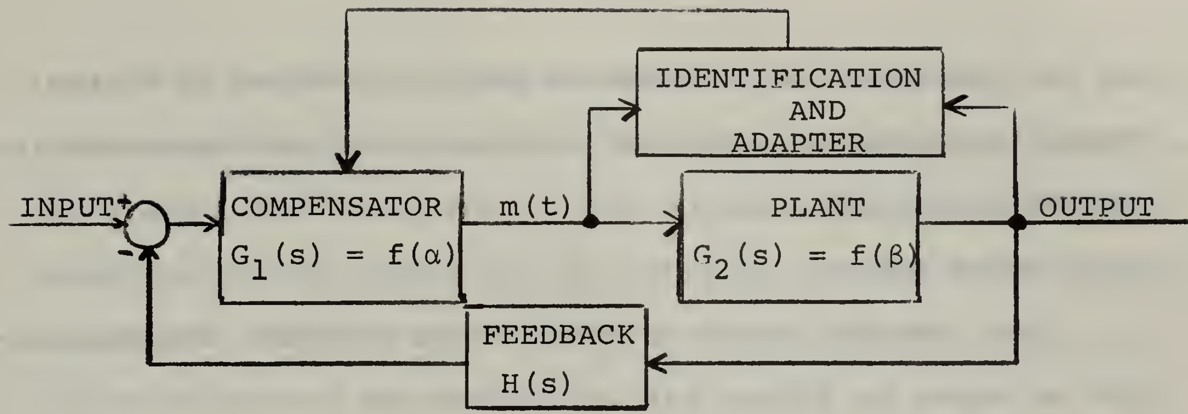

Figure 5-3. Self-Adaptive Control System

and the output of the plant in order to identify the parameter β. When a change in β has been detected the "identification and adapter" block adjusts the variable compensation block parameter α such that the system is returned to its original specification. The design of the "identification and adapter" block is the critical portion of any self-adaptive system and involves two requirements:

(1) Some method must be devised for identifying the change within the system's plant,

(2) A means of adjustment must be provided which is capable of counteracting the effects of the parameters variations.

Identification of any change in the parameters of a plant have been covered in references 14 and 15. References 16 and 17 are concerned with methods for providing adjustment schemes. This chapter shows how associated polynomials theory may be used in the development of a "compensating ratio" between two variables parameters of a feedback control system. It shows that a change in a variable parameter

in the plant may be compensated for by a change in another variable parameter such that dominant root performance or behavior is controlled.

5.2: Basic Theory

The transfer function of a simple feedback control system as shown in figure 5-1 is written as:

$$T(s) = \frac{C}{R}(s) = \frac{G(s)}{1+HG(s)} \qquad (5\text{-}1)$$

The denominator of equation 5-1 when equated to zero is the characteristic equation of the system. This equation is a polynomial of degree n and may be represented as:

$$W(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0 = 0 \qquad (5\text{-}2)$$

By the fundamental theorem of algebra, the characteristic equation has n roots. Hence, equation 5-2 may be written:

$$W(s) = (s+p_1)(s+p_2)\ldots(s+p_n) = 0 \qquad (5\text{-}3)$$

Consider the coefficients of the characteristic equation, i.e., $a_0, a_1, \ldots, a_n$. Assume any two variable parameters α and β appear linearly in any number of coefficients, i.e.,

$$a_j = b_j\alpha + c_j\beta + d_j \qquad (5\text{-}4)$$

$$: j = 0,1,\ldots,n$$

where b_j, c_j, and d_j are real constants. Note that the coefficient a_j is a linear function of α and β. This representation of variable parameters in the coefficients of the characteristic equation was first introduced by D. D. Siljak[18] and later expanded by G. J. Thaler, et al.[19], in the development of parameter plane theory.

From general polynomial theory, the change in any root is determined by the change in the coefficients of the defining polynomial and vice-versa. The roots of the characteristic equation can be considered as functions of the characteristic equation's coefficients. Hence the differential $p_i(a_0, a_1, \ldots, a_n)$ can be expressed as:

$$dp_i = \frac{\partial p_i}{\partial a_0} da_0 + \frac{\partial p_i}{\partial a_1} da_1 + \ldots + \frac{\partial p_i}{\partial a_n} da_n \tag{5-5}$$

From equation 5-4, the coefficients, a_j: $j = 0, 1, \ldots, n$, are themselves functions of the variable parameters α and β. Hence the differential of any coefficient is:

$$da_j = \frac{\partial a_j}{\partial \alpha} d\alpha + \frac{\partial a_j}{\partial \beta} d\beta \tag{5-6}$$

Substituting equation 5-6 into equation 5-5:

$$dp_i = \frac{\partial p_i}{\partial a_0}\left[\frac{\partial a_0}{\partial \alpha} d\alpha + \frac{\partial a_0}{\partial \beta} d\beta\right] + \frac{\partial p_i}{\partial a_1}\left[\frac{\partial a_1}{\partial \alpha} d\alpha + \frac{\partial a_1}{\partial \beta} d\beta\right]$$

$$+ \ldots + \frac{\partial p_i}{\partial a_n}\left[\frac{\partial a_n}{\partial \alpha} d\alpha + \frac{\partial a_n}{\partial \beta} d\beta\right] \tag{5-7}$$

From equation 5-4:

$$\left.\begin{aligned} \frac{\partial a_j}{\partial \alpha} &= b_j \\ & \qquad (j = 0, 1, \ldots, n) \\ \frac{\partial a_j}{\partial \beta} &= c_j \end{aligned}\right. \tag{5-8}$$

Substituting equation 5-8 into equation 5-7 and then factoring $d\alpha$ and $d\beta$;

$$dp_i = \left[b_0 \frac{\partial p_i}{\partial a_0} + b_1 \frac{\partial p_i}{\partial a_1} + \ldots + b_n \frac{\partial p_i}{\partial a_n} \right] d\alpha +$$

$$\left[c_0 \frac{\partial p_i}{\partial a_0} + c_1 \frac{\partial p_i}{\partial a_1} + \ldots + c_n \frac{\partial p_i}{\partial a_n} \right] d\beta \qquad (5\text{-}9)$$

From associated polynomial theory, the variation of a polynomial's roots with the variation of its coefficients is expressable as:

$$\frac{\partial p_i}{\partial a_j} = -\frac{p_i^{\,j}}{F'(p_i)} \qquad j = 0,1,\ldots,n \qquad (5\text{-}10)$$

Using the above equation, the differential of any root, p_i, may be written as:

$$dp_i = \frac{b_0 + b_1 p_i + \ldots + b_n p_i^{\,n}}{-F'(p_i)} \, d\alpha +$$

$$\frac{c_0 + c_1 p_i + \ldots + c_n p_i^{\,n}}{-F'(p_i)} \, d\beta \qquad (5\text{-}11)$$

In this chapter the term "compensating ratio" is used for the mathematical relationship between two variable parameters such that the system's dominant performance or behavior is controllable. Since the dominant root of a feedback control system may be either real or a complex conjugate pair, four different cases of compensating adaption are possible. These cases are:

(1) The dominant root is real and is to be held constant,

(2) The dominant roots are a complex conjugate pair and are to be held constant,

(3) The dominant roots are a complex conjugate pair and the damping coefficient is to be held constant,

(4) The dominant roots are a complex conjugate pair and the damped resonant or natural frequency is to be held constant.

5.3: Case (1): The Dominant Root is Real and is to be Held Constant

When the dominant root, p_i, is real and is to be held constant for changes in the parameters α and β, then the differential of the root must be equal to zero, i.e.,

$$dp_i = 0.$$

With $dp_i = 0$, equation 5-11 reduces to:

$$\frac{d\alpha}{d\beta} = -\frac{c_0+c_1p_i+c_2p_i^2+\ldots+c_np_i^n}{b_0+b_1p_i+b_2p_i^2+\ldots+b_np_i^n} \qquad (5\text{-}12)$$

Where $d\alpha/d\beta$ is the definition of "compensating ratio". The constants, b_j and c_j are determined from the coefficients of the characteristic equation. Since for this case, p_i is real, the compensating ratio is:

$$\frac{d\alpha}{d\beta} = -\text{Constant} \qquad (5\text{-}13)$$

<u>Example 5-1</u>

Figure 5-4 is a block diagram of a type one, unity feedback control system. The system's characteristic equation is third order and contains the variable parameters α and β. The characteristic equation of the system shown in figure 5-4 is:

$$s^3 + (P_1+\beta)s^2 + (P_1\beta+K)s + \alpha K = 0 \qquad (5\text{-}14)$$

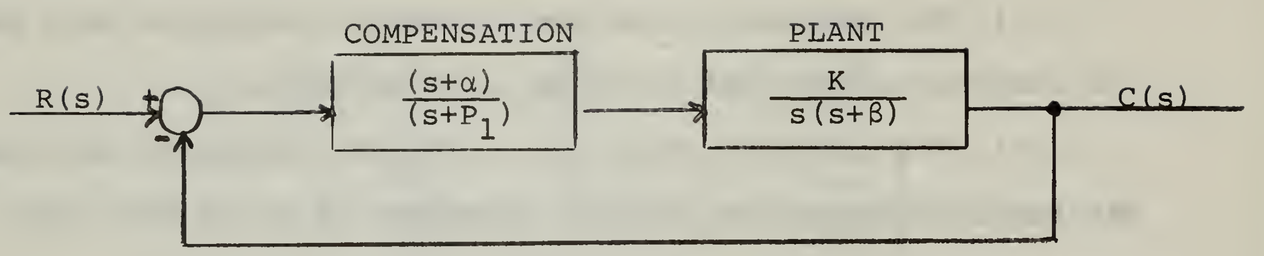

Figure 5-4: Block Diagram Example 5-1 System

Choosing the components and gain factor such that the system is stable let:

$$P_1 = 12;\ \alpha = 3;\ \beta = 1;\ K = 210.$$

The characteristic equation is now:

$$s^3 + 13s^2 + 222s + 630 = 0$$

which factors to:

$$(s+3.318)(s+4.84+j12.9)(s+4.84-j12.9) \qquad (5\text{-}15)$$

Without any type of compensation or self-adaption, the behavior of the system is dependent upon changes in the parameter β. Table 5-1 lists various values of β and the corresponding factors of the systems characteristic equation. For convenience these values are plotted in figure 5-5.

Table 5-1: System Roots With a Variable Parameter

β Values	Factors of Characteristic Equation		
.01	(s+3.49)	(s+4.26+j12.73)	(s+4.26-j12.73)
.1	(s+3.47)	(s+4.31+j12.75)	(s+4.31-j12.75)
1.0	(s+3.318)	(s+4.84+j12.9)	(s+4.84-j12.9)
2.0	(s+3.1)	(s+5.4 +j13.0)	(s+5.4 -j13.0)
10.0	(s+2.2)	(s+9.9 +j13.7)	(s+9.9 -j13.7)

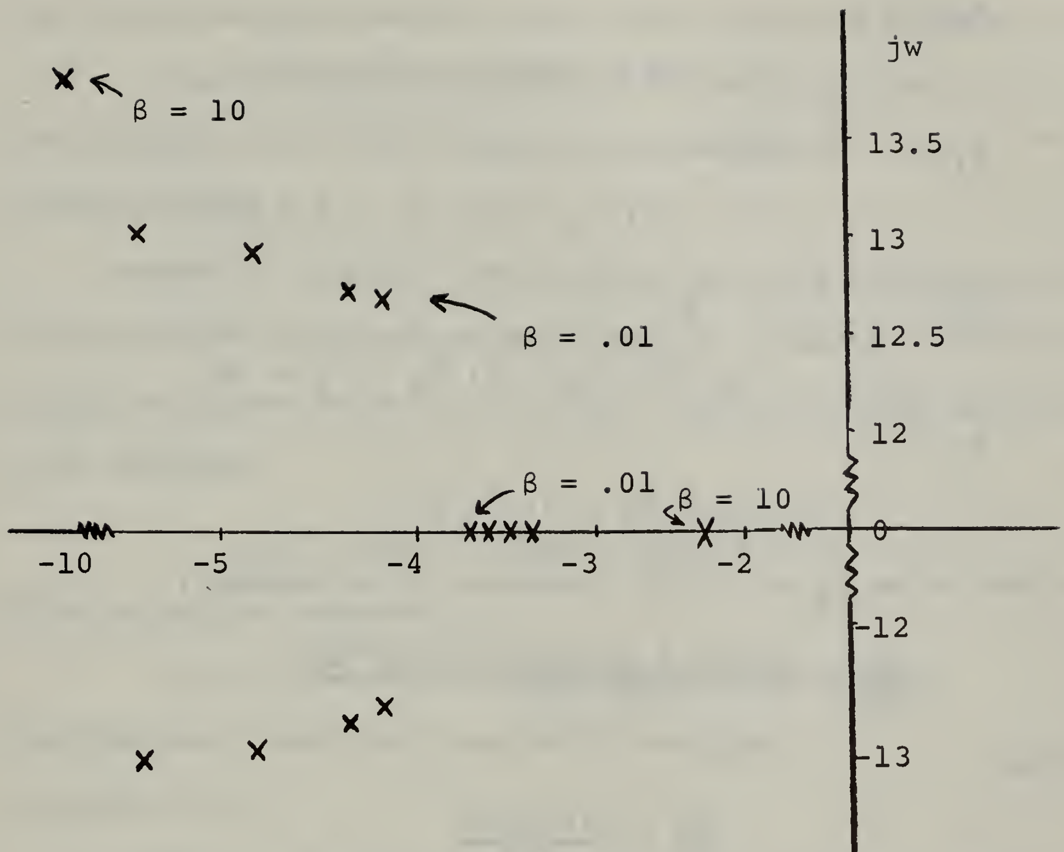

Figure 5-5: System Poles With a Variable Parameter

Assuming the desired operating point for the system is that position specified by the value β = 1, equation 5-15 gives the factors of the characteristic equation. Note that the dominant root of the system is $p_i = -3.318$. The compensation network has been added to provide adaption so that any variations in β can be cancelled by the adjustment of the parameter α insofar as the dominant root performance is concerned.

Without specifying any change in α or β, evaluate the constant term of equation 5-13, i.e.,

$$\frac{c_0+c_1p_i+c_2p_i^2+\ldots+c_np_i^n}{b_0+b_1p_i+b_2p_i^2+\ldots+b_np_i^n} \qquad (5\text{-}17)$$

$a_0 = K\alpha$

$$b_0 = K = 210; \; c_0 = d_0 = 0$$

$a_1 = \quad + P_1\beta + K$

$$b_1 = 0; \; c_1 = 12; \; d_1 = K = 210$$

$a_2 = \quad + \beta + P_1$

$$b_2 = 0; \; c_2 = 1; \; d_2 = P_1 = 12$$

$a_3 = 1$

$$b_3 = c_3 = 0; \; d_3 = 1$$

With the value $p_i = -3.318$, equation 5-12 becomes:

$$\frac{d\alpha}{d\beta} = -\frac{0+12(-3.318)+1(-3.318)^2+0}{210 + 0}$$

reducing

$$\frac{d\alpha}{d\beta} = +\frac{28.7936}{210}$$

hence:

$$d\alpha = +.13711 d\beta \tag{5-18}$$

From the results of equation 5-18, for any change in the plant's parameter β, in order to keep the root $p_i = -3.318$ constant, α must be changed $(.137)d\beta$.

Situation (1):

Assume βchanges from a value of 1 to a new value, $\beta = 2$. As the change in β is plus one, α must be changed (.137)(1) to a value $\alpha' = +3.137$. These new values of α and β change the characteristic equation to:

$$s^3 + 14s^2 + 234s + 658.79 = 0$$

By a root solving method, the above equation factors to:

$$(s+3.3179)(s+5.34+j13.04)(s+5.34-j13.04)$$

The dominant root has been held constant.

Situation (2):

Assume β changes from a value of 1 to a value $\beta = 10$. The variable parameter α must then be changed $(.137)(9) = 1.233$ to a new value $\alpha' = 4.233$. The new characteristic equation becomes:

$$s^3 + 22s^2 + 330s + 889.14 = 0.$$

This equation factors to:

$$(s+3.3174)(s+9.34+j13.45)(s+9.34-j13.45)$$

The dominant root has been held constant.

Situation (3):

Assume β changes from a value of 1 to $\beta = .9$. This 10 percent decrease forces α to change $-.0137$ to a new value $\alpha' = 2.9873$. The characteristic equation becomes:

$$s^3 + 12.9s^2 + 220.8s + 627.123 = 0$$

which factors to:

$$(s+3.318)(s+4.79+j12.88)(s+4.79-j12.88).$$

The dominant root has been held constant.

<u>Example 5-2</u>

Given a type '0', unity feedback system as shown in figure 5-6. The system characteristic equation is third order and is given by:

$$s^3 + (P_1+P_2+\beta)s^2 + [P_1P_2+KK_1+(P_1+P_2)\beta]s + KK_1\alpha + P_1P_2\beta = 0 \tag{5-19}$$

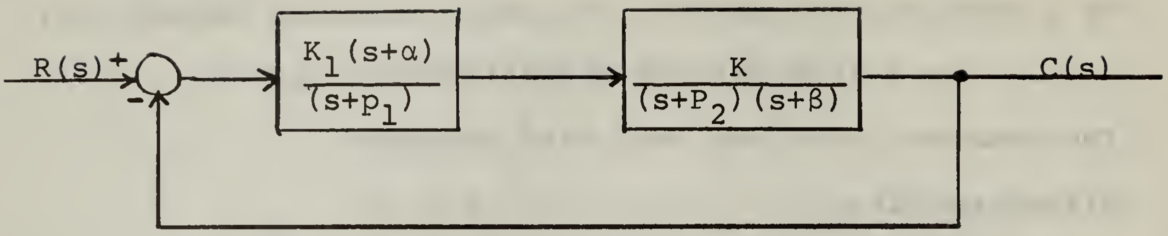

Figure 5-6: Block Diagram Example 5-2 System.

Let the values of the various components be assigned such that the system is stable. These values are:

$$P_1 = 2;\ P_2 = 3;\ KK_1 = 4;\ \alpha = 5;\ \beta = 6.$$

The characteristic equation with these values substituted becomes:

$$s^3 + 11s^2 + 40s + 56 = 0.$$

The factors of the above equation are:

$$(s+5.716)(s+2.642+j1.678)(s+2.642-j1.678).$$

The results of the Case (1) compensating ratio can be used such that for any variation in the parameter β, one will be able to vary the parameter α and keep the real root, $p_i = -5.716$ constant. The procedure to be followed can be outlined into three steps.

Step (1): Determine the value of the constant terms in the coefficients of the characteristic equation:

$$a_0 = KK_1\alpha + P_1P_2\beta$$

$$b_0 = KK_1 = 4;\ c_0 = P_1P_2 = 6;\ d_0 = 0$$

$$a_1 = P_1P_2 + KK_1 + (P_1+P_2)\beta$$

$$b_1 = 0;\ c_1 = P_1 + P_2 = 5;\ d_1 = P_1P_2 + KK_1 = 10 \tag{5-20}$$

$a_2 = P_1 + P_2 + \beta$

$$b_2 = 0;\ c_2 = 1;\ d_2 = P_1 + P_2 = 5$$

$a_3 = 1$

$$b_3 = c_3 = 0;\ d_3 = 1 \qquad (5\text{-}20)$$

Step (2): Using the values found in equation 5-20 and the real root $p_i = -5.716$, evaluate the compensating ratio equation 5-12.

$$\frac{d\alpha}{d\beta} = -\frac{6+5(-5.716)+1(-5.716)^2}{4}$$

$$\frac{d\alpha}{d\beta} = -\frac{9.0926}{4} \text{ or } d\alpha = -2.273d\beta \qquad (5\text{-}21)$$

Step (3): Identify the change in the parameter β and adjust the value of α according to equation 5-21. The real root will remain constant.

Assume β changes from a value of +6 to a new value of 5.3. $d\beta = -.7$ and the change in α must be:

$$d\alpha = -2.273(-.7) = 1.59$$

Substitute the new value of $\alpha = 5 + 1.59$ into the characteristic equation along with the new value of β. With $\alpha = 6.59$ and $\beta = 5.3$, the new characteristic equation becomes:

$$s^3 + 10.3s^2 + 36.5 + 58.2 = 0.$$

This equation factors to:

$$(s+5.693)(s+2.31+j2.214)(s+2.31-j2.214).$$

5.4: Case (2): The Dominant Roots are a Complex Conjugate Pair and are to be Held Constant

When the dominant roots of the characteristic equation are a complex conjugate pair, an increment in one parameter,

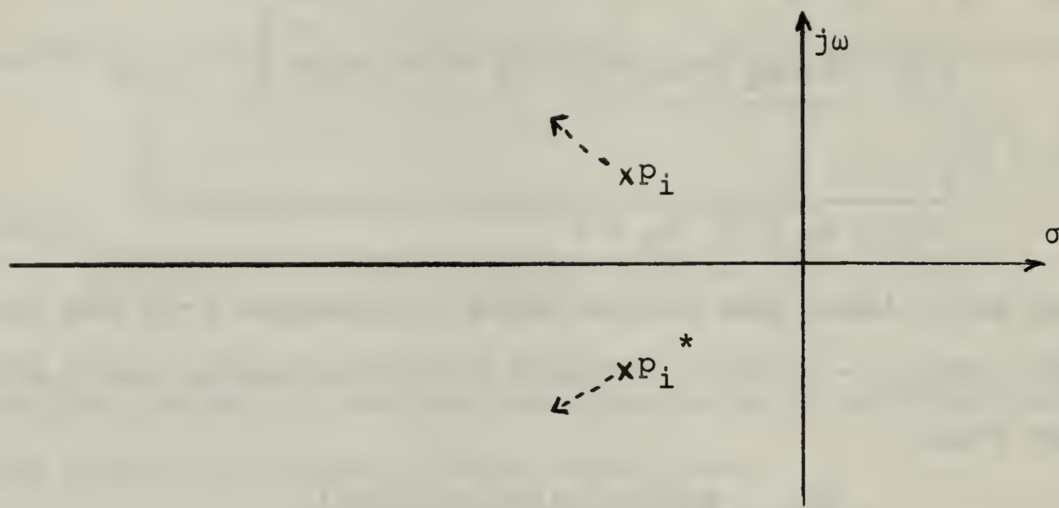

Figure 5-7. Change in Root Location With a Change in a System Parameter

such as β, changes the root locations as shown in figure 5-7.

Although any new root location may occur, it is important to note that the two roots will always be complex conjugates. The behavior of complex conjugate roots is far more complex than the real root case presented earlier. Instead of a simple plus or minus variation as in the real root location, an uncontrollable parameter variation in the system's plant will cause the complex dominant roots, p_i and p_i*, to change both in their real and imaginary components. In this section, it is proven that with two variable parameters in a system, the dominant root locations can be returned to their original positions by corrective compensation in only very trivial cases.

For convenience equation 5-11 is repeated below. This equation shows that the differential of any root, p_i, is a function of the change in any two variable parameters α and β.

$$dp_i = \frac{b_0+b_1p_i+\ldots+b_np_i^n}{-F'(p_i)}\,d\alpha + \frac{c_0+c_1p_i+\ldots+c_np_i^n}{-F'(p_i)}\,d\beta \tag{5-11}$$

where: b_j and c_j, $j = 0,1,\ldots,n$ are constants determined by the system,

: α and β are real parameters, initially fixed in value but subject to variations.

If p_i is complex and is desired to be held constant, then the differential of the root p_i must be set equal to zero, i.e.,

$$dp_i = 0 \; .$$

Equation 4-11 then reduces to:

$$\frac{d\alpha}{d\beta} = \frac{c_0+c_1p_i+\ldots+c_np_i^n}{b_0+b_1p_i+\ldots+b_np_i^n} \tag{5-22}$$

As the parameters α and β are specified to be real by definition, then the right-hand side of equation 5-22 must be real to have any meaning. There must be restrictions on the system parameters in order to make the compensating ratio, equation 5-22, equal to a real constant.

To find these restrictions, assume the numerator of the compensating ratio with a complex root, p_i, may be combined into the general format: $m + jn$. Likewise assume that the denominator is expressable as: $x + jy$. Equation 5-21 may now be written:

$$\frac{d\alpha}{d\beta} = -\frac{m+jn}{x+jy} \tag{5-23}$$

Separating the above equation into its real and imaginary components:

$$\frac{d\alpha}{d\beta} = -\frac{(m+jn)}{(x+jy)} \cdot \frac{(x-jy)}{(x-jy)} \tag{5-24}$$

or

$$\frac{d\alpha}{d\beta} = -\left[\frac{mx+ny}{x^2+y^2} + j\frac{nx-my}{x^2+y^2}\right] \tag{5-25}$$

Equation 5-25 will be real if and only if nx = my. The compensating ratio is:

$$\frac{d\alpha}{d\beta} = -\frac{m}{x} = -\frac{n}{y} \tag{5-26}$$

The numerator and the denominator of the compensating ratio must be constant factors of each other. As will be shown, when p_i is complex the above conditions imply pole-zero cancellation. This result is expressed in the following theorem.

THEOREM 5-1

For the n^{-th} order feedback control system shown in figure 5-8 with variable system parameters, α and β, there will exist a closed-loop root, p_i, whose location remains invariant for all increments in α or β such that the compensating ratio, $d\alpha/d\beta$, is equal to a real constant provided, if and only if:

(a) the root of interest, p_i, is real or

(b) the variable parameter α equals the variable parameter β which implies a zero-pole cancellation, a trivial case.

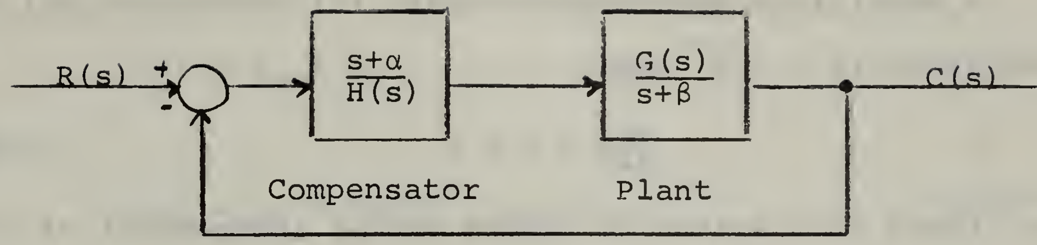

Figure 5-8. n^{-th} Order Feedback Control System

Proof:

For convenience, the compensating ratio for any n^{-th} order feedback control system is repeated:

$$\frac{d\alpha}{d\beta} = -\frac{c_0+c_1p_i+\ldots+c_np_i^n}{b_0+b_1p_i+\ldots+b_np_i^n} \tag{5-22}$$

If the above ratio is to be equal to a real constant 'K', then equation 5-22 reduces to:

$$K(b_0+b_1p_i+\ldots+b_np_i^n) = -(c_0+c_1p_i+\ldots+c_np_i^n) \tag{5-27a}$$

equating to zero:

$$(b_0K+c_0) + (b_1K+c_1)p_i + \ldots + (b_nK+c_n)p_i^n = 0 \tag{5-27b}$$

A sufficient condition for part (a) of Theorem 5-1 is that equation 5-27 is always satisfied when the root of interest, p_i, is real. This can be shown to be true by inspection of the compensating ratio, equation 5-22. The parameters α and β are by definition, real parameters. The constant terms, b_j and c_j, are also defined to be real numbers. If p_i is real, then equation 5-22 is a ratio of two real numbers, hence dα/dβ is equal to a real constant. If this real constant is equal to 'K', then equation 5-27 must always be satisfied for any real root, p_i.

A sufficient condition for part (b) of Theorem 4-1 follows since if $\alpha = \beta$, then

$$\frac{s+\alpha}{s+\beta} = K = 1 \qquad (5\text{-}28)$$

The closed loop system in figure 5-8 is independent of α and β.

The necessary condition for the proof of Theorem 5-1 can be shown with the aid of figure 5-9 where the real factor, k, has been added for generality.

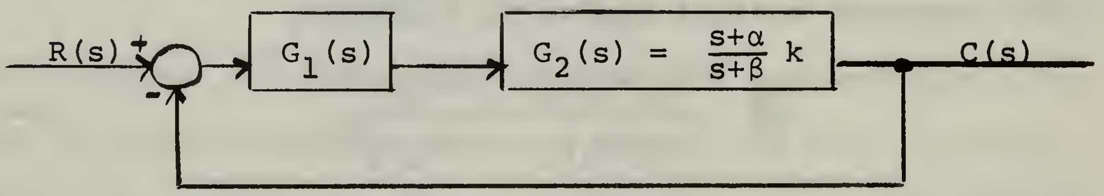

Figure 5-9. Modified Closed Loop Feedback Control System

The closed-loop control system of figure 5-8 has been transformed into the modified control system of figure 5-9 by the following modifications:

$G_1(s) = G(s)/H(s)$, the fixed portion of the closed loop system.

$G_2(s) = k(s+\alpha)/(s+\beta)$, the adjustable portion of the closed loop system.

The characteristic equation is:

$$k\,\frac{G(s)}{H(s)}\,\frac{(s+\alpha)}{(s+\beta)} + 1 = 0 \qquad (5\text{-}29a)$$

or

$$H(s)(s+\beta) + kG(s)(s+\alpha) = 0 \qquad (5\text{-}29b)$$

or

$$s[kG(s)+H(s)] + \alpha kG(s) + \beta H(s) = 0 \qquad (5\text{-}29c)$$

Assume that G(s) and H(s) may be expressed as:

$$G(s) = g_0 + g_1 s + \ldots + g_{n-1} s^{n-1} \qquad (5\text{-}30a)$$

and

$$H(s) = h_0 + h_1 s + \ldots + h_{n-1} s^{n-1} \qquad (5\text{-}30b)$$

Substituting equations 5-30a and 5-30b into equation 5-29c, the characteristic equation becomes:

$$\begin{aligned}(kg_0\alpha+h_0\beta) &+ (kg_1\alpha+h_1\beta+kg_0+h_0)s + \ldots \\ &+ (kg_{n-1}\alpha+h_{n-1}\beta+kg_{n-2}+h_{n-2})s^{n-1} \\ &+ (kg_{n-1}+h_{n-1})s^n \qquad (5\text{-}31)\end{aligned}$$

Equation 5-31 is the n^{-th} order characteristic equation of the feedback control system of figure 5-8, which is written as:

$$a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + a_n s^n = 0 \qquad (5\text{-}32)$$

From the introduction, the coefficients of the characteristic equation are expressable as:

$$a_j = b_j\alpha + c_j\beta + d_j \qquad (5\text{-}4)$$

$$j = 0,1,\ldots,n$$

Equating the coefficients of equations 5-31 and 5-32 yields:

$$a_0 = kg_0\alpha + h_0\beta = b_0\alpha + c_0\beta + d_0 \qquad (5\text{-}32a)$$

$$a_1 = kg_1\alpha + h_1\beta + (kg_0+h_0) = b_1\alpha + c_1\beta + d_1 \qquad (5\text{-}32b)$$

. . .

$$a_{n-1} = kg_{n-1}\alpha + h_{n-1}\beta + (kg_{n-2}+h_{n-2}) = b_{n-1}\alpha + c_{n-1}\beta + d_{n-1} \qquad (5\text{-}32c)$$

$$a_n = (kg_{n-1}+h_{n-1}) = b_n\alpha + c_n\beta + d_n \qquad (5\text{-}32d)$$

Equating the constant terms:

$$
\begin{array}{llll}
b_0 = kg_0 & c_0 = h_0 & d_0 = 0 & \text{(5-34a)} \\
b_1 = kg_1 & c_1 = h_1 & d_1 = kq_0 + h_0 & \text{(5-34b)} \\
\cdots & & & \\
b_{n=1} = kg_{n-1} & c_{n-1} = h_{n-1} & d_{n-1} = kg_{n-2} + h_{n-2} & \text{(5-34c)} \\
b_n = 0 & c_n = 0 & d_n = kg_{n-1} + h_{n-1} & \text{(5-34d)}
\end{array}
$$

With the above values of b_j and c_j, equation 5-27b becomes:

$$(kg_0K+h_0) + (kg_1K+h_1)p_i + \ldots + (kg_{n-1}K+h_{n-1})p_i^{n-1} = 0 \qquad \text{(5-35)}$$

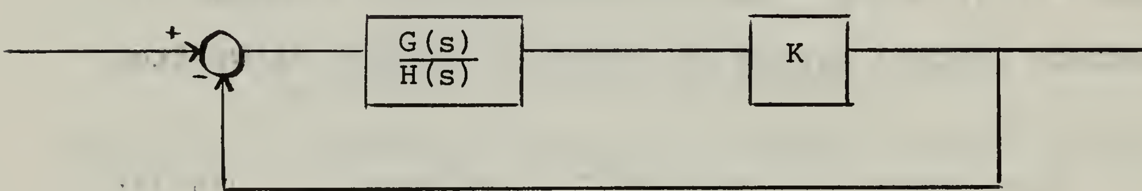

Figure 5-10. $(n-1)^{th}$ Order Feedback Control System

Figure 5-10 shows an $(n-1)^{th}$ order closed loop feedback control system. The characteristic equation of this system is:

$$K\frac{G(s)}{H(s)} + 1 = 0 \qquad \text{(5-36a)}$$

or

$$KG(s) + H(s) = 0 \qquad \text{(5-36b)}$$

If G(s) and H(s) are represented by equations 5-30, then equation 5-36b can be written as:

$$(g_0K+h_0) + (g_1K+h_1)s + \ldots + (g_{n-1}K+h_{n-1})s^{n-1} = 0 \quad \text{(5-37)}$$

Comparing equation 5-35 with equation 5-37, it follows that the root, p_i, must also be a root of the system shown in figure 5-10, that is,

$$kK \frac{G(s)}{H(s)} + 1 = 0 \tag{5-38}$$

Since p_i is a root of the original system shown in figure 5-8, then it also follows that:

$$\frac{(p_i+\alpha)}{(p_i+\beta)} \frac{G(s)}{H(s)} + 1 = 0 \tag{5-39}$$

Equating equations 5-38 and 5-39:

$$\frac{(p_i+\alpha)}{(p_i+\beta)} = kK \tag{5-40}$$

Solving for the value of the root, p_i:

$$(p_i+\alpha) = kK(p_i+\beta) \tag{5-41}$$

or

$$p_i = \frac{\beta-\alpha}{1-kK} \tag{5-42}$$

From equation 5-42, the root of interest, p_i, is always real if kK is not equal to one. If kK is equal to one, then p_i is indeterminate, however from equation 5-28, $kK = 1$ implies $\alpha = \beta$ or zero-pole cancellation, a trivial case of compensation.

5.5: Case (3): The Dominant Roots are Complex Conjugates and the Damping Coefficient is to be Held Constant

Since the damping coefficient, $\zeta\omega_n$, is to be held constant for variations in the plant's parameter β, let the real part of either dominant complex conjugate root be held constant, i.e.,

$$\mathcal{R}_e[p_i] = \text{CONSTANT} \tag{5-43}$$

With the $\mathcal{R}_e[p_i]$ equal to a constant value the real part of the differential of p_i must equal zero, or

$$\mathcal{R}_e[dp_i] = 0 \tag{5-44}$$

With the above condition, the real part of equation 5-11 is written:

$$\mathcal{R}_e[dp_i] = 0 = \mathcal{R}_e\left[\frac{b_0+b_1p_i+\ldots+b_np_i^n}{-F'(p_i)}\,d\alpha\right] + \mathcal{R}_e\left[\frac{c_0+c_1p_i+\ldots+c_np_i^n}{-F'(p_i)}\,d\beta\right] \tag{5-45}$$

hence:

$$\mathcal{R}_e\left[\frac{b_0+b_1p_i+\ldots+b_np_i^n}{-F'(p_i)}\,d\alpha\right] = -\mathcal{R}_e\left[\frac{c_0+c_1p_i+\ldots+c_np_i^n}{-F'(p_i)}\,d\beta\right] \tag{5-46}$$

Evaluating the terms in the parenthesis of equation 5-46, let

$$m + jn = \frac{b_0+b_1p_i+\ldots+b_np_i^n}{-F'(p_i)} \tag{5-47}$$

and

$$u + jv = \frac{c_0+c_1p_i+\ldots+c_np_i^n}{-F'(p_i)} \tag{5-48}$$

Using the above notation, equation 5-16 is rewritten as:

$$\mathcal{R}_e[md\alpha+jnd\alpha] = -\mathcal{R}_e[ud\beta+jvd\beta] \tag{5-49a}$$

or

$$md\alpha = ud\beta \tag{5-49b}$$

The compensating ratio for a constant damping coefficient is therefore equal to:

$$\frac{d\alpha}{d\beta} = -\frac{u}{m} \tag{5-50}$$

Example 5-3

Using the system of example 5-2, the characteristic equation is:

$$s^3 + (P_1+P_2+\beta)s^2 + [P_1P_2+KK_1+(P_1+P_2)\beta]s + KK_1\alpha + P_1P_2\beta = 0$$

with the parameter values:

$$P_1 = 2;\ P_2 = 3;\ KK_1 = 4;\ \alpha = 5;\ \beta = 6.$$

The numerical characteristic equation is

$$s^3 + 11s^2 + 40s + 56 = 0$$

which factors to:

$$(s+5.716)(s+2.642+j1.678)(s+2.642-j1.678).$$

The parameter β will vary and again must be compensated for by a change in the parameter α, such that the damping coefficient, $\zeta\omega_n$, is held constant. The damping coefficient will be held constant if equation 5-50 is satisfied, therefore the following procedure is recommended.

Step (1): Determine the value of the constant terms in the coefficients of the characteristic equation.

This step has been performed in example 5-2 and equation 5-20 is repeated for convenience.

$b_0 = 4$	$c_0 = 6$	$d_0 = 0$
$b_1 = 0$	$c_1 = 5$	$d_1 = 10$
$b_2 = 0$	$c_2 = 1$	$d_2 = 5$
$b_3 = 0$	$c_3 = 0$	$d_3 = 1$

Step (2): Form the real part of equations 5-47 and 5-48.

Denominator:

$$F(s) = s^3 + 11s^2 + 40s + 56$$

$$F'(s) = 3s^2 + 22s + 40$$

$$p_i = -2.64 - j10.35$$

$$F'(p_i) = 3(-2.64-j1.68)^2 + 22(-2.64-j1.68) + 40$$

$$F'(p_i) = -5.64 - j10.35$$

Equation 5-47:

$$m + jn = \frac{b_0+b_1p_i+b_1p_i^2}{-F'(p_i)}$$

$$= \frac{4}{5.64+j10.35}$$

$$m = \frac{4(5.64)}{(5.64)^2+(10.35)^2} \qquad \underline{m = .163}$$

Equation 5-48:

$$u + jv = \frac{c_0+c_1p_i+c_2p_i^2}{-F'(p_i)}$$

$$= \frac{-3.1+j.43}{5.64+j10.35} = \frac{3.13 \underline{/7.9}}{11.8 \underline{/61.4}}$$

$$\underline{u = -.094}$$

The compensating ratio is the negative ratio of u over m or

$$d\alpha = +.573d\beta$$

Step (3): Identify the change in the parameter β and adjust.

Assume β changes from a value of 6 to a value of 5.3.

The change in β is:

$$d\beta = -.7$$

$$d\alpha = (-.7)(.573)$$

$$d\alpha = -.4011$$

The parameter α must change to 5 - .4011 = 4.5989. The new characteristic equation will be:

$$s^3 + 10.3s^2 + 36.5s + 50.2 = 0.$$

This equation factors to:

$$(s+5.01)(s+2.64+j1.74)(s+2.64-j1.74).$$

By inspection, the real part of the complex conjugate pair has been kept constant.

5.6: Case (4): The Dominant Roots are a Complex Conjugate Pair and the Damped Resonant Frequency is to be Held Constant

In a similar manner as the damping coefficient, $\zeta\omega_n$, was held constant in case (3), the damped resonant or natural frequence, ω_d, will be held constant by keeping the imaginary part of either dominant complex root constant, i.e.,

$$\mathcal{I}_m[p_i] = \text{CONSTANT}$$

With the Imaginary part of p_i held constant, the imaginary part of the differential of p_i must equal zero, or

$$\mathcal{I}_m[dp_i] = 0 \qquad (5\text{-}51)$$

Setting the imaginary part of the differential of p_i equal to zero, equation 5-11 becomes:

$$\mathcal{I}_m\left[\frac{b_0+b_1p_i+\ldots+b_np_i^n}{-F'(p_i)}\,d\alpha\right] = \mathcal{I}_m\left[\frac{c_0+c_1p_i+\ldots+c_np_i^n}{-F'(p_i)}\,d\beta\right] \qquad (5\text{-}52)$$

Inspecting the terms in the parenthesis as done in case (3), equations 5-47 and 5-48 are evaluated. The compensating ratio for a constant damped resonant frequency is found to be:

$$\frac{d\alpha}{d\beta} = -\frac{v}{n} \qquad (5\text{-}53)$$

where

$$v = \mathcal{I}_m \left[\frac{c_0 + c_1 p_i + \ldots + c_n p_i^n}{-F'(p_i)} \right] \qquad (5\text{-}54)$$

and

$$n = \mathcal{I}_m \left[\frac{b_0 + b_1 p_i + \ldots + b_n p_i^n}{-F'(p_i)} \right] \qquad (5\text{-}55)$$

Example 5-4

Using the system of example 5-2 and 5-3, the characteristic equation is:

$$s^3 + (P_1+P_2+\beta)s^2 + [P_1P_2+KK_1+(P_1+P_2)\beta]s + KK_1\alpha + P_1P_2\beta = 0$$

Substituting in the values of the parameters as before, the characteristic equation becomes:

$$s^3 + 11s^2 + 40s + 56 = 0$$

which factors to:

$$(s+5.716)(s+2.64+j1.68)(s+2.64-j1.68).$$

In this example the damped resonant frequency will be held constant by the following procedure as indicated in the above case.

Step (1): Determine the value of the constant terms in the coefficients of the characteristic equation.

Repeating equation 5-20 for convenience:

$b_0 = 4$	$c_0 = 6$	$d_0 = 0$
$b_1 = 0$	$c_1 = 5$	$d_1 = 10$
$b_2 = 0$	$c_2 = 1$	$d_2 = 5$
$b_3 = 0$	$c_3 = 0$	$d_3 = 1$

Step (2): Form the imaginary parts of equation 5-47.

Denominator:

$$F(s) = s^3 + 11s^2 + 40s + 56$$

$$F'(s) = 3s^2 + 22s + 40$$

$p_i = -2.64-j10.35$: the dominant root of interest. As before in example 5-2:

$$F'(p_i) = -5.64-j10.35.$$

Equation 5-47:

$$m + jn = \frac{4}{5.64+j10.35}$$

$$= \frac{4}{11.8\underline{/61.5}}$$

$$\underline{n = -.298}$$

Equation 5-48:

$$u + jv = \frac{-3.1+j.43}{5.64+j10.35}$$

$$= .266\underline{/110°46'}$$

$$\underline{v = .246}$$

Step (3): Determine the compensating ratio according to equation 5-53.

$$\frac{d\alpha}{d\beta} = -\frac{.246}{-.298}$$

$$d\alpha = +.825d\beta$$

Step (4): Identify the change in the parameter β and adjust.

If β changes from a value of 6 to a new value $\beta = 5.3$, the change in β is:

$$d\beta = -.7$$

$$d\alpha = (-.7)(.825)$$

$$d\alpha = -.577$$

The parameter α must change to

$$5 - .577 = 4.423 .$$

The new characteristic equation will be:

$$s^3 + 10.3s^2 + 36.5s + 49.4 = 0.$$

This equation factors to:

$$(s+4.94)(s+2.68+j1.683)(s+2.68-j1.683).$$

By inspection the imaginary part of the dominant root has been held constant by proper use of the compensating ratio, equation 5-53.

CHAPTER VI

Conclusions

In this thesis, a purely mathematical property of all polynomials of degree n, (called the associated polynomials of the defining polynomial) has been applied to the fields of circuits and systems. The properties of associated polynomials have been investigated and examples given that demonstrate that these polynomials provide useful results. It has been shown that associated polynomials may be used in a new method for accomplishing a partial fraction expansion including the troublesome case of repeated roots of high order of multiplicity. Associated polynomial theory has been used to study the relationship between variations of polynomial coefficients and polynomial roots. The theorems developed in Chapter Three that show that the variation of the coefficients of a polynomial with respect to the variation of its roots or vice-versa is determined by associated polynomials have important engineering applications. They are immediately used in a root solving procedure and to express sensitivity coefficients in a new analytical form.

The most significant result of this thesis is the investigation of the theory of compensating parameters as applied to self-adaptive control systems. A mathematical relationship between two variable parameters remains invariant such that the system's dominant root performance, has been derived. In addition, a theorem of fundamental importance in adaptive

control theory has been presented that affects the basic theory of complex-conjugate dominant root adaptive compensation.

As a suggestion for further research, it is recommended that an actual computer program be written that performs the partial fraction expansion of any ratio of polynomials. In addition, it is believed that the results of Chapter Three have logical extensions to a derivation of an analytical expression for a root-locus plot. This analytical expression could then be programmed onto a digital computer. Lastly, the results of Chapter Five can be extended to variable parameters that are not linearly related to the coefficients of the characteristic equation, i.e., the variable parameters would appear in the form:

$$a_j = b_j\alpha + c_j\beta + h_j\alpha\beta + d_j \qquad (6\text{-}1)$$

or:

$$a_j = b_{j2}\alpha^2 + b_{j1}\alpha + h_j\alpha\beta + c_{j1}\beta + c_{j2}\beta^2 + d_j \qquad (6\text{-}2)$$

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APPENDIX A

Horner's Method

A.1: Introduction

A complete historical survey of associated polynomials is given in reference 1. In this reference it is noted that what has been labeled associated polynomials are sometimes referred to as Horner polynomials. This appellation comes about by three processes involving associated polynomials that are attributed to Horner[4]. These processes are:

(1): The nested evaluation of polynomials;

(2): Synthetic division;

(3): A method of approximating the roots of polynomial equations.

Each of the three processes above are commonly called Horner's method or rule[20]. As is shown, the method of approximating the roots of a polynomial directly involves synthetic division.

A.2: The Nested Evaluation of Polynomials

Associated polynomials may be used in the evaluation of polynomials by a process called "nesting". In order to evaluate the polynomial,

$$F(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0 : \quad a_n = 1 \qquad (1\text{-}1)$$

for any value of s, there is required 2n + 1 multiplications and n additions. Using associated polynomials there is required only n multiplications and n additions. This process of evaluation is performed by the following method:

$$F(s=u) = a_n u^n + a_{n-1} u^{n-1} + \ldots + a_0 : \quad a_n = 1 \qquad (A\text{-}1)$$

but

$$F(u) = A_0(u) \tag{1-12}$$

where:

$$A_0(u) = uA_1(u) + a_0 \tag{1-9n}$$

$$A_1(u) = uA_2(u) + a_1 \tag{1-9n-1}$$

$$\cdot \ \cdot \ \cdot$$

$$A_{n-1}(u) = uA_n(u) + a_{n-1} \tag{1-9b}$$

and since:

$$A_{n+1}(u) = 0 \tag{1-7}$$

$$A_n(u) = uA_{n+1}(u) + a_n = a_n \tag{1-9a}$$

Hence F(u) is evaluated using only n multiplications and n additions.

A.3: Synthetic Division

The equation:

$$\frac{F(s)}{s-u} = \frac{A_0(u)}{s-u} + \sum_{j=1}^{n} A_j(u) s^{j-1} \tag{1-15}$$

states that the quotient and remainder after division by a polynomial of the form s - u may be calculated recursively. The nature of the recursion is such that one may work with detached coefficients. By detached coefficients is meant an abbreviation of the ordinary multiplication and division processes used in ordinary algebra. The coefficients alone with their signs are used in the process. The powers of the variable occurring in the various terms are understood only from the order in which the coefficients are written. Missing

powers are assumed to be present with zero coefficients. The process of synthetic division is best understood by the following example and summary.

Example A-1

Given the polynomial:

$$F(s) = As^3 + Bs^2 + Cs + D \tag{A-2}$$

And the divisor polynomial:

$$s - a \tag{A-3}$$

The usual process of division is as follows:

$$\begin{array}{rl}
As^3 + Bs^2 + Cs + D \,\big/ & \underline{\;s-a\;} \\
\underline{As^3 - Aas^2} \qquad\qquad & As^2+Es+F \\
Es^2 + Cs \qquad & \\
\underline{Es^2 - Eas} \qquad & \\
Fs + D & \\
\underline{Fs - Fa} & \\
D + Fa = R &
\end{array}$$

Or:

$$\frac{As^3+Bs^2+Cs+D}{s-a} = (As^2+Es+F) + \frac{R}{s-a} \tag{A-4}$$

Comparing with equation 1-15:

$$A_0(a) = R \tag{A-5}$$

and

$$A_j(a)s^{j-1} = (As^2+Es+F): \quad j = 1,2,\ldots,n \tag{A-6}$$

Using synthetic division, the process of division is reduced by only using the coefficients according to the following scheme.

$$\begin{array}{cccc|l}
A & B & C & D & \underline{/+a} \\
 & Aa & Ea & Fa & \\
\hline
A & E & F & R &
\end{array}$$

Notice that the first three numbers of the last line, A, E, and F, are the coefficients of s^2, s, and 1, in the quotient, and R is the remainder. The process may be expanded to any n^{th} order polynomial.

As a summary, synthetic division may be performed according to the following rule.

(1): To divide F(s) by a polynomial of the form, s - a, arrange F(s) into descending powers of s.

(2): Write the coefficients of F(s) on a horizontal line, in the order which corresponds to the arrangement specified in (1). If any power of s is missing in F(s), supply that power with a zero coefficient.

(3): Multiply the first coefficient A by a, write the product below the second coefficient B and add. Multiply this sum, E, by a, write the product below the third coefficient, D, and add.

(4): Proceed in this way until all of the places in the third row except the first are filled up, and write down the first coefficient, A, of F(s) in the first place of the third row. The last number of the third row will be the remainder, and the other numbers in the third row will be the coefficients of the quotient obtained when F(s) is divided by s - a.

A.4: Approximation of the Roots of a Polynomial Equation

A method for approximating the real roots of an algebraic equation first appeared in a paper read by W. G. Horner before the Royal Society in 1819[4]. This process involves associated polynomials in that repeated use of synthetic

division is used to expedite the work of substituting values for the variable. The essential steps[21] of Horner's method of approximation are as follows:

(1): Using curve plotting, the theorem of continuous fractions*, etc., isolate a positive root between two successive integers. If the equation has only negative roots, transform it to one whose roots are the negatives of those of the given equation.

(2): Transform the equation into an equation whose roots are decreased by the lesser of the integers between which the root lies, by the substitution $x' = x-a$. The root of the new equation will lie between zero and unity.

(3): Isolate the root of the new equation between successive tenths.

(4): Transform the last equation into an equation whose roots are decreased by the smaller of these tenths and isolate the root of this equation between hundredths. Continue this process to one decimal place more than the place to which the answer is to be correct.

The root sought is then the total amount by which the roots of the original equation were reduced, namely, the lesser integer, plus the lesser tenth, plus the lesser hundredth, etc., the last decimal being rounded off to make the result accurate to the desired decimal place.

---

*If $P(x)$ is a polynomial with real coefficients and of degree $n>1$, and if a and b are real numbers such that $P(a)$ and $P(b)$ have opposite signs, then the polynomial $P(x)$ has at least one root between a and b.(22)

Example A-2

Find to two decimal places that root of the equation:

$$s^3 + s^2 - 12s + 11 = 0 \qquad \text{(A-7)}$$

which lies between 2 and 3.

The root in question will be of the form 2.abc...,
where a, b, c, ... stand for its decimal figures.

(1): Using synthetic division, first transform equation A-7 into an equation whose roots are those of equation A-7 diminished by 2.

$$\begin{array}{rrrrl}
1 & 1 & -12 & 11 & \underline{/2} \\
 & & 6 & -12 & \\
\hline
1 & 3 & -6 & -1 & \\
 & 2 & 10 & & \\
\hline
1 & 5 & 4 & & \\
 & 2 & & & \\
\hline
1 & 7 & & &
\end{array}$$

The transformed equation is:

$$s^3 + 7s^2 + 4s - 1 = 0 \qquad \text{(A-8)}$$

Equation A-8 will have the root .abc... lying between 0 and 1.

(2): Substituting the values s = .1, .2, .3, ... into equation A-8, one finds that:

$$f(.1) < 0$$

and

$$f(.2) > 0$$

hence equation A-8 has a root between .1 and .2, so that:

$$a = 1 \qquad \text{(A-9)}$$

(3): Diminishing the roots of equation A-8 by .1 using synthetic division:

$$
\begin{array}{lllll}
1 & 7 & 4 & -1 & \underline{/.1} \\
 & .1 & .71 & .471 & \\
\hline
1 & 7.1 & 4.71 & -.529 & \\
 & .1 & .72 & & \\
\hline
1 & 7.2 & 5.43 & & \\
 & .1 & & & \\
\hline
1 & 7.3 & & &
\end{array}
$$

The transformed equation is:

$$s^3 + 7.3s^2 + 5.43s - .529 = 0 \qquad \text{(A-10)}$$

which has a root of the form .0bc . . .

(4): Substituting the values s = .01, .02, ... into equation A-10, one finds that:

$$g(.08) < 0$$

and

$$g(.09) > 0 \quad .$$

Hence equation A-10 has a root between .08 and .09 so that:

$$b = 8 \ .$$

From equation A-9 and A-11, the root of equation A-7 which lies between 2 and 3 to two decimal places is:

$$s = 2.18 \ .$$

The process could be continued indefinitely, thus permitting one to compute the root of equation A-7 between 2 and 3 to any specified number of decimal places.

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13 ABSTRACT

In this thesis, the theory of associated polynomials is applied to linear circuits and systems. Associated polynomials are defined and used in the presentation of a computer algorithm suitable for accomplishing a partial fraction expansion including repeated roots. Variations of the coefficients of a polynomial are related to the variation of its roots (and vice-versa) by associated polynomials. These results are used in a root solving process and to express sensitivity coefficients in a new analytical form. Lastly, a theory of compensating parameter adjustments is developed with applications to self-adaptive systems. It is shown that by compensating adjustments of any two parameters in the system which are linearly related to the coefficients of the characteristic equation, adaptive compensation is possible for a real root of the closed loop transfer function only and that for complex roots, the real and imaginary parts may be kept invarient but not both.

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SELF-ADAPTIVE CONTROL SYSTEMS						
VARIABLE PARAMETERS						
CONTROL SYSTEMS						